Spiral–Defect Chaos: Swift-Hohenberg model versus Boussinesq equations

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Spiral–defect chaos (*SDC*) in Rayleigh–Bénard convection is a well established spatio–temporal complex pattern, which competes with stationary rolls near onset of convection. The characteristic properties of SDC are accurately described on the basis of the standard three–dimensional Boussinesq–equations. As a much simpler and attractive two–dimensional model for SDC generalized Swift–Hohenberg (*SH*) equations have been extensively used in the literature from the early beginning. Here we show that the SH–description of SDC has to be considered with care, especially regarding its long–time dynamics. For parameters used in previous SH-simulations SDC occurs only as a transient in contrast to the experiments and the rigorous solutions of the Boussinesq equations. The small–scale structure of the vorticity field at the spiral cores, which might be crucial for persistent SDC, is presumably not perfectly captured in the SH–model.

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Convection in a horizontal fluid layer heated from below, known as Rayleigh-Bénard convection (*RBC*), is one of the best studied examples of pattern forming systems [1–3]. At threshold convection rolls bifurcate and remain stable in a fairly wide parameter range, coined as the Busse–Balloon. Thus the recent observation of spiral–defect chaos (*SDC*) in a parameter regime where it competes with rolls was rather surprising [4,5]. The complex spatio–temporal dynamics of SDC involves rotating spirals, targets, dislocations etc. Most of characteristic properties of SDC are well reproduced in high precision ab initio solutions of the standard Boussinesq equations [6–8] in three spatial dimensions. According to the experiments and the numerical solutions SDC is a robust generic state of thermal convection observed in rectangular, square and circular cells as well [9,10,6,3].

Our general understanding of the universal aspects of pattern formation has been significantly promoted by the analysis of two–dimensional models like the various types of Ginzburg–Landau and Swift–Hohenberg equations [1,11,12]. This applies also to SDC where simulations of generalized Swift–Hohenberg (*SH*) equations [15,16] have provided important insight into the underlying mechanism [13,20,14]. Nevertheless, one should remain open to possible limitations of such models. On the concrete example of SDC we show in this paper, that the long-time dynamics of the SH equations might be problematic.

In the following we discuss simulations of SDC in a set of widely used SH equations, which couple two real fields ψ and ζ [13,14].

$$\Big[\partial_t + g_m \mathbf{U}\cdot\nabla\Big]\psi = \Big[\varepsilon - (1+\Delta)^2\Big]\psi - \psi^3, \tag{1a}$$

$$\Big[\tau_\zeta\partial_t - \mathcal{P}(\eta\nabla^2 - c^2)\Big]\Delta\zeta = \Big[(\partial_y\psi)\partial_x - (\partial_x\psi)\partial_y\Big]\Delta\psi. \tag{1b}$$

Here $\psi(\mathbf{r},t)$ describes the planar spatial variations of convection patterns (e.g. the temperature field), which consist locally of convection-roll patches. $\zeta(\mathbf{r},t)$ is a velocity potential determining the mean flow $\mathbf{U} = (\partial_y\zeta, -\partial_x\zeta)$. The control parameter $\varepsilon = 2.78\,(\Delta T - \Delta\, T_c)/\,\Delta\, T_c$ serves as a dimensionless measure for the applied temperature difference ΔT across the fluid layer. [17]. The time is scaled in such a way that a time lapse of $t = 5$ in Eqs.(1) corresponds to $\sim 1\,t_v$, with the common vertical diffusion $t_v \sim O(\text{sec})$ in experiments.

Any curvature of the rolls produces a vertical vorticity field $-\Delta\zeta(\mathbf{r},t)$ (see Eq. (1b)), which increases with decreasing Prandtl number $\mathcal{P}$. In contrast to the claims expressed in several papers by Gunton and coworkers (see e.g. [27], only the dominant term $\sim c^2$ on the left-hand side of Eq. (1b) can be directly traced back to the Boussinesq equations. The two other terms $\propto \tau_g, \eta$, respectively, are in principle phenomenological (see the discussion in [?]). In Eq. (1a) the relevance of ζ is controlled by the coupling constant g_m. The value of g_m can be found to be $g_m = 12.2$ for $c^2 = 2$ and $Pr = 1$ by comparing with the known zig-zag stability boundary of convection rolls [18].

The coupling to the mean flow, which becomes more important either at small $\mathcal{P}$ or large g_m is crucial for persistent SDC. In the limit of large Prandtl numbers $\mathcal{P}$ where ζ is hardly excited the dynamics of ψ becomes purely relaxational and approaches a low dimensional stationary state of the corresponding Lyapunov functional. Note, however, that any strongly disordered pattern before it equilibrates generates virtually instanteneously a strong, long-range mean-flow $\mathbf{U}$ according to Eq. (1b and can thus easily lead to transient SDC, even if Pr is not small.

In our simulations we have chosen the same set of parameters as in the previous works [13,27,?], namely $c^2 = 2, g_m = 50, \tau_\zeta = \eta = Pr = 1, \epsilon = 0.7$. Mostly we consider an aspect ratio of $\Gamma = L/2d = 32$ where L denotes the lateral extension of the cell and d its thickness. At first we have performed simulations in a square domain with periodic boundary conditions in order to avoid an artificial bias from the sides. Starting from random initial conditions yields a typical snapshot as shown in Fig. (1a) at $800t_v$. This initial pattern compares well with planforms already shown in Refs. [13,14] at the same time lapse; it has also great similarity to characteristic SDC snapshots observed continuously in experiments [4,5] or during numerical solutions of the fundamental Boussinesq equations [6].

However, when continuing the runs beyond $8000t_v$ the scenario changes qualitatively and the pattern coarsens towards a "big spiral", which rotates slowly about a nearly immobile center. Only at the boundaries of the spiral one finds remnants of the previous persistent generation and annihilation of small spirals. For $\mathcal{P} \approx 1$ the coarsening to big spirals is neither observed in experiments nor during simulations of the Boussinesq equations.

The transient behaviour of SDC followed by coarsenig to a big spiral reminds to recent experiments at $\mathcal{P} = 4$ [9]. After a sudden quench strongly disorderd pattern developed which due to the strong vorticity field let to the SDC transient. Afterwards SDC coarsend to a big spiral as well which eventually disintegrated after a long time towards a stationary pattern. Apparently for $\mathcal{P} = 4$ the vorticity field is to weak to sustain SDC.

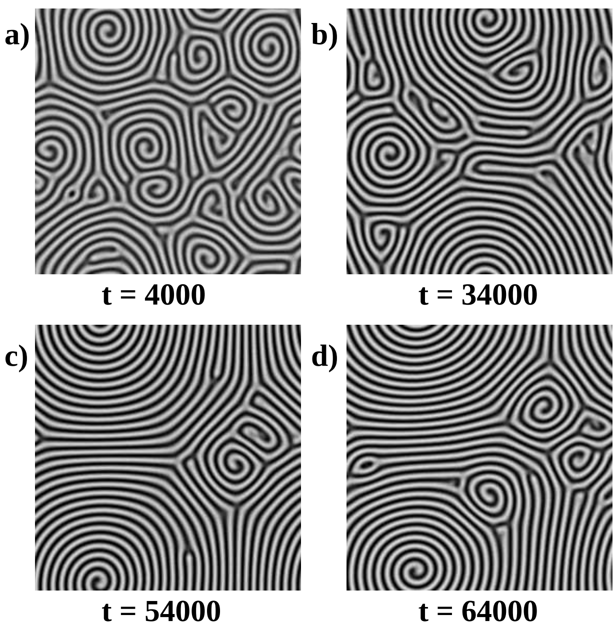

FIG. 1. The field $\psi(\mathbf{r}, t)$ is plotted a different times after starting with random initial conditions. The parameters are $\Gamma = 32$ (aspect ratio), $\varepsilon = 0.7$, $g_m = 50$, $c^2 = 2$, $\mathcal{P} = 1$ and $\tau_v = 1$.

Unlike the experiments and the solutions of the Boussinesq equations the SDC attractor in the SH-simulations shows a remarkable sensitivity to the boundary conditions. This feature becomes evident, when the SH-equations, Eq.(1), are simulated on a circular domain, similar as in the original work in Ref. [13]. Initially we observe a similar coarsening process as in the case of a square domain with periodic boundary conditions leading to a big spiral about the cell center which is rather long living (on the average $500 - 1000t_v$). Possibly because of the focus instability [23,24] the spiral core moves off center and the spiral arms may be compressed and react in a sudden process by the generation of dislocation pairs, inevitably associated with a strong vorticity field. The dislocation tips wind up in a dynamics, that has been loosely described as "invasive chaos" by Cross et. al [14,22] . During that period one observes SDC that coarsens again to a quite big spiral which becomes again unstable and so on. The periodic dynamics due to the generation of dislocations in compressed roll patches is typical for circular cells and has been described in other comparable situations as well (see e.g. [25]).

The difference between the latter scenario of "intermittent SDC" and persistent generic SDC is apparent from Fig.2, where we compare the normalized heat current j_n as obtained from simulations of Eqs. (1) for a circular geometry with j_n obtained for simulations of the full Boussinesq equations [6] in a circular geometry too. In both cases the heat current is only shown on a small representative time window taken out of a very long runs which lasted up to $t_v = 40000$. In the upper panel we see rather rare but violent events, when the coarsened big spirals breaks up at the downward spike [28]. In contrast, the heat current in the the Boussinesq case (lower panel), shows only small fluctuations.

A rather small aspect ratio of $\Gamma = 32$ is sufficient to obtain persistent SDC in the experiments [3] and has facilitated the extensive simulations of the Boussinesq equations described before, Nevertheless we have tested in some runs, whether persistent SDC possibly requires a larger Γ in the SH simulations. However, we observed no qualitative changes for $\Gamma = 64$ (the other parameters in Fig. 1 remained fixed), though not unexpectedly the coarsening set in at later times.

With respect to some deficiencies of the SH alluded to above we have several speculations. The fact that the Busse balloon is not correctly reproduced in the SH-description [6,19,?] and that long SDC periods require a four times larger g_m value than the theoretical one (see above and [?]), might be of minor importance. However, already when Manneville introduced generalized SH equations [15] he discussed in detail the intricate roll of the mean flow and some very long transients before his simulations settled down to a steady attractor. In addition, the general SH-equations rest a long wave–length approximation for $\zeta(\mathbf{r}, t)$. On constrast, the pronounced short–scale structures in the vorticity which exist for instance at a spiral corei (see e.g. Fig. 18 in [?]) are not systematically captured. They might in reality permanently "stir" the system keeping a persistent weak turbulence alive.

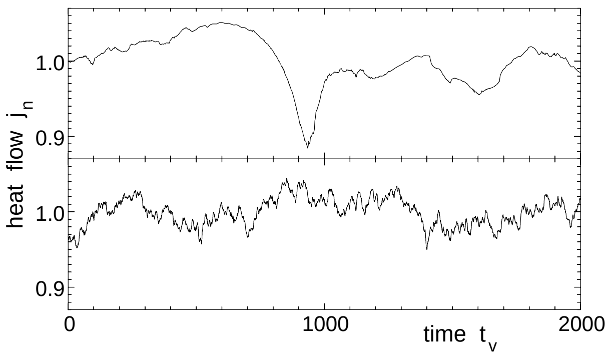

FIG. 2. The convective heat flux $j(t) = \frac{1}{S}\int_S d^2\mathbf{r}\, u^2(\mathbf{r},t)$ normalized to its time average $< j >$, i.e. $j_n = j/ < j >$, is shown for the SH–model (upper part) with the same parameters as in Fig.1 and for the Boussinesq equations [6] (lower part).

In conclusion, for many purposes generalized SH-models are certainly very valuable tools to study the SDC scenario, even if it would exist only as a long transient. However, our investigations shed some light on the general problem of understanding the long–time behavior of hydrodynamic systems by using SH models. Accordingly, their application to coarsening studies [26] or to the analysis of statistical properties of SDC [27] might be questionable.

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