SPACE-DEPENDENT ORDER PARAMETER IN CIRCULAR COUETTE FLOW TRANSITIONS

G. PFISTER and I. REHBERG
Institute of Applied Physics, University of Kiel, Germany

Received 10 December 1980
Revised manuscript received 12 March 1981

We show that in circular Couette flow (i) for the transition to steady Taylor-vortex flow and (ii) for the transition to time-dependent wavy-vortex flow a space-dependent order parameter obeys a Ginzburg–Landau-type equation. Our measurements allow extrapolation of the critical Reynolds numbers to infinitely long cylinders.

We present here the first detailed measurements of the axial dependence of the order parameter both for the transition from circular Couette flow to stationary Taylor-vortex flow and the subsequent transition to time-periodic wavy-vortex flow. For both transitions, the end plates influence the flow. Cole [1] and Kuznetsov et al. [2] observed that vortex cells build up gradually from the ends of the cylinder. Thus the estimation of a critical Reynolds number must be more or less subjective [3]. Similarly, wavy vortices depend strongly on cylinder length to gap ratio [1] and on boundary conditions.

While the first transition is a continuous process and therefore not an instability [3] the transition to the wavy-vortex state remains sharp and gives definite critical Reynolds numbers [4,5].

Much theoretical work has been done concerning these transitions, but usually these calculations refer to infinite annulus length only or employ unrealistic boundary conditions [6].

The order parameter concept allows a parametrization and theoretical description of hydrodynamic instabilities [7,8]. For the case of stationary Taylor vortices, Donnelly [9], Donnelly and Schwarz [10], and more recently Gollub and Freilich [11] showed that the order parameter dependences on Reynolds number and time follow the theoretical predictions. They did not take into account any spacial dependence of the order parameter as Wesfreid et al. [12] did for the Bénard instability. The latter authors obtained the predicted dependence of the order parameter on Rayleigh number, time and space.

Taking the space dependence into account, the evolution of the order parameter v is governed by the time-dependent Ginzburg–Landau equation [13]:

$$\tau_0 \, \partial V/\partial t = \epsilon V - V^3/V_0^2 + \xi_0^2 \, \partial^2 V/\partial x^2, \qquad (1)$$

with $\epsilon = (\mathrm{Re} - \mathrm{Re_c})/\mathrm{Re_c}$, where Re is the Reynolds number [‡1] and $\mathrm{Re_c}$ the critical Reynolds number for infinitely long cylinders, τ_0 is the linear amplification rate, V_0 a normalization constant such that $\sqrt{\epsilon}\, V_0 = V$ for cylinders of infinite length and ξ_0 the length unit ≈ cell dimension.

This model holds for the most stable wavelength of Taylor vortices only. Changing the wavelength by 5% we found changes in the critical Reynolds number of the onset of the wavy mode up to 15%. This phenomenon was clearly described by Benjamin and Mullin [4,5].

For the stationary case ($\partial/\partial t = 0$) considere here, eq. (1) may be integrated. It is convenient to introduce the (maximal or minimal) amplitude $V = V_\mathrm{m}$ which is realized at midlength $x = 0$ of the cylinder.

The solutions are [14]:

$$\epsilon V_0^2 \leqslant V_\mathrm{m}^2/2: \quad V = V_\mathrm{m}\,\mathrm{nc}((x/\xi_0)(V_\mathrm{m}^2/V_0^2 - \epsilon)^{1/2}|m), \qquad (2a)$$

‡1 $\mathrm{Re} = r_1(r_2 - r_1)\Omega_1/\nu$, $\Omega_1 = 2\pi \times$ rotation rate of inner cylinder, ν is the kinematic viscosity.

with

$$m = (V_m^2 - 2\epsilon V_0^2)/(2V_m^2 - 2\epsilon V_0^2);$$

$$V_m^2/2 \leqslant \epsilon V_0^2 \leqslant V_m^2:$$

$$V = V_m \,\mathrm{dc}((x/\xi_0) V_m/(V_0 \sqrt{2})|m), \tag{2b}$$

with

$$m = (2\epsilon V_0^2)/V_m^2 - 1;$$

$$V_m^2 \leqslant \epsilon V_0^2: \quad V = V_m \,\mathrm{cd}((x/\xi_0)(\epsilon - V_m^2/2V_0^2)^{1/2}|m), \tag{2c}$$

with

$$m = V_m^2/(2\epsilon V_0^2 - V_m^2).$$

The functions nc, dc and cd are jacobian elliptic functions depending on argument and parameter.

In our experiments, several grades of silicon oil were confined between an inner rotating cylinder (r_1 = 11.56 mm) and an outer stationary cylinder (r_2 = 22.82 mm) yielding a gap width of 11.26 mm, which was uniform to within 1% over the entire length. Top and bottom end plates were stationary and could be adjusted up to a maximum separation of L = 404 mm. The fluid was thermostated to within 0.05 K corresponding to an uncertainty of 0.1% in the Reynolds number. The local velocity was measured by a real fringe laser Doppler anemometer described elsewhere [15].

(i) Measured radial velocity versus annulus height

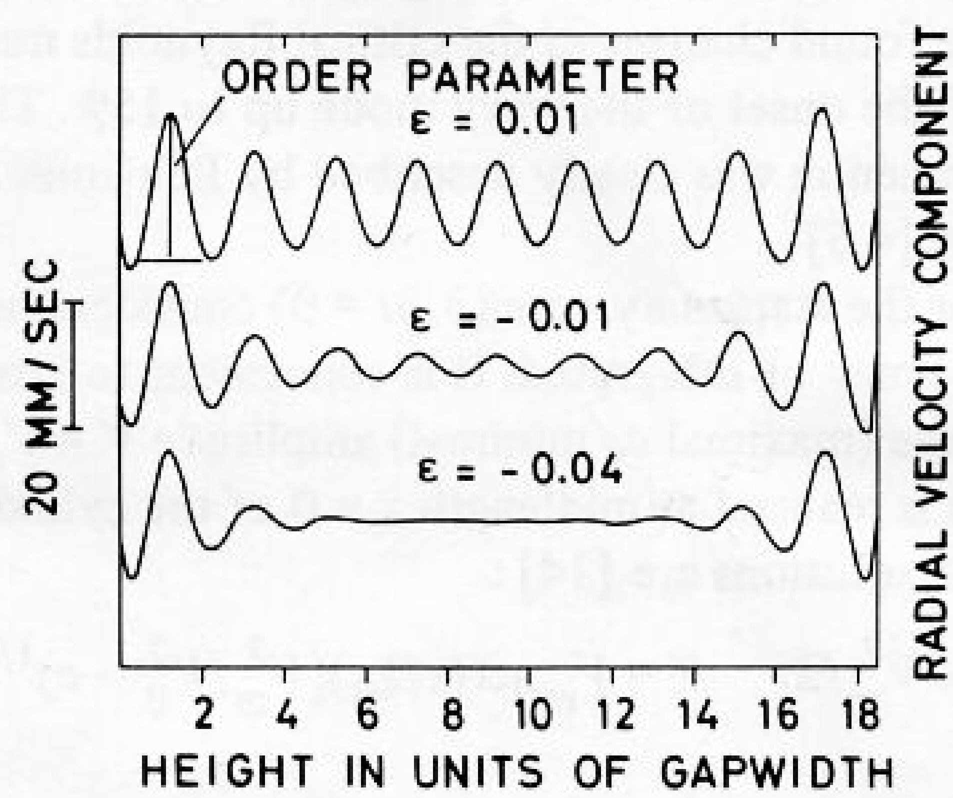

Fig. 1. Measured radial velocity profiles for three rotation frequencies of the inner cylinder. Note that the end vortices induce vortices even below $\epsilon = 0$.

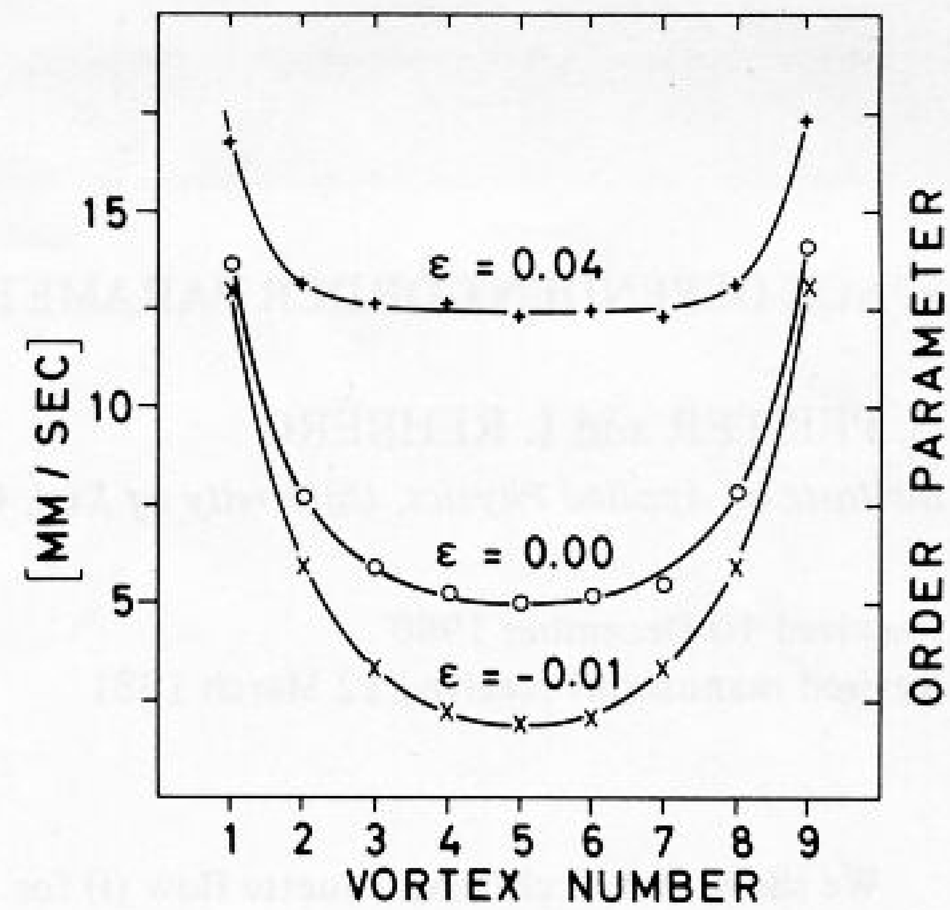

Fig. 2. Measured space-dependent order parameter for three values of ϵ. The solid lines are least-squares fittings according to eqs. (2).

for the stationary Taylor vortex flow in shown in fig. 1 for three different rotation rates. In this case the height to gap ratio was 18.5, and the total number of vortex pairs was 9. The figure suggests that the end (Ekman) vortices induce (Taylor) vortices even below $\epsilon = 0$. For comparison with theory, the maximum amplitude within each vortex pair is taken as the order parameter, which therefore assumes discrete values only. By fitting the measured values of the order parameter to the appropriate function (2a) or (2b), a value of the parameter m is obtained. An example of such a fit is shown in fig. 2. From m as defined in eq. (2a) or (2b) and the measured value V_m one computes $\sqrt{\epsilon}\, V_0$, the order parameter for the infinitely long annulus. A plot of a sequence of such values $\sqrt{\epsilon}\, V_0$, or rather its square ϵV_0^2, versus rotation rate yields the critical rotation rate or Reynolds number that would be valid for the infinitely long geometry as shown in fig. 3.

Our experimental value of Re_c = 68.0 agrees well with those obtained theoretically by Kirchgässner [16] and Davey [17] which are 68.2 and 68.19, respectively. The main experimental error is due to the viscosity measurement, which is about 1%.

(ii) The second instability is the wavy, time-periodic mode, first reported in detail by Coles [18]. If the gap width $r_2 - r_1$ is comparable to r_1, the first azimuthal wave number is $n = 1$ observed by flow visualization and rate correlation velocimetry [19]. To

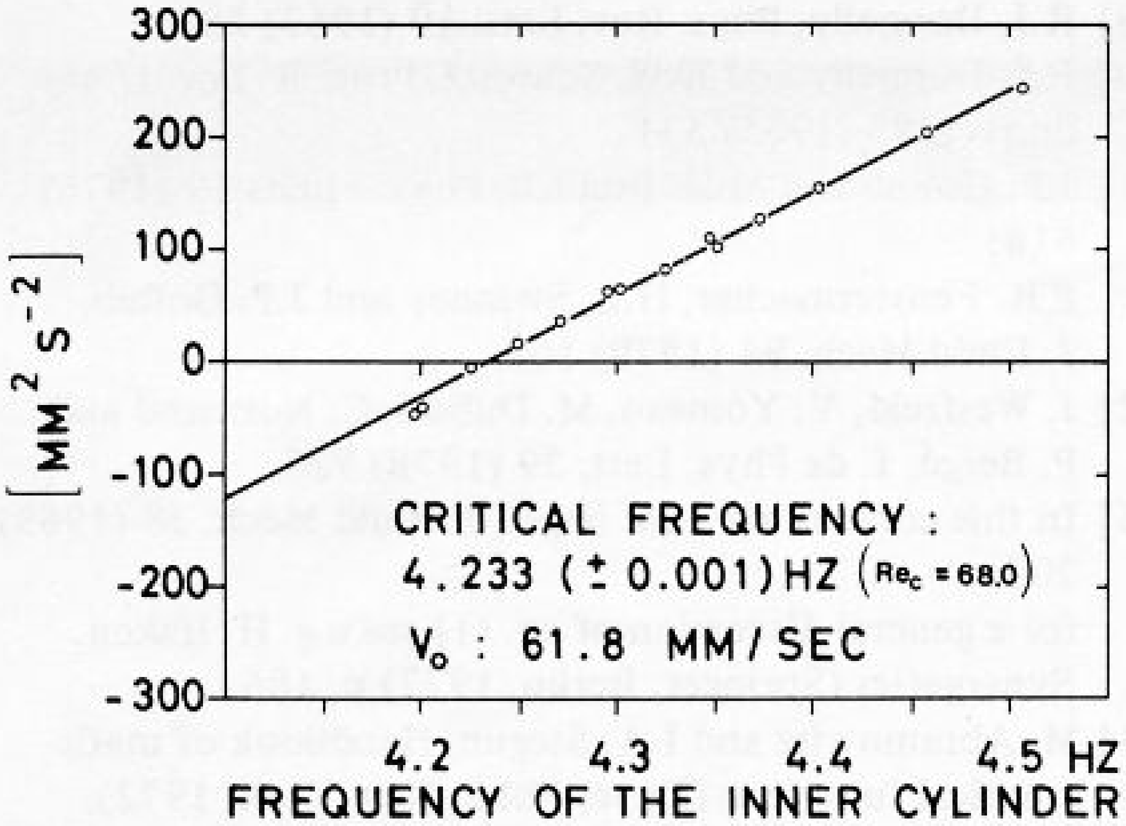

Fig. 3. Plot of ϵV_0^2 data (as obtained from previous fits) versus rotation frequency. Intercept yields critical frequency of infinite geometry.

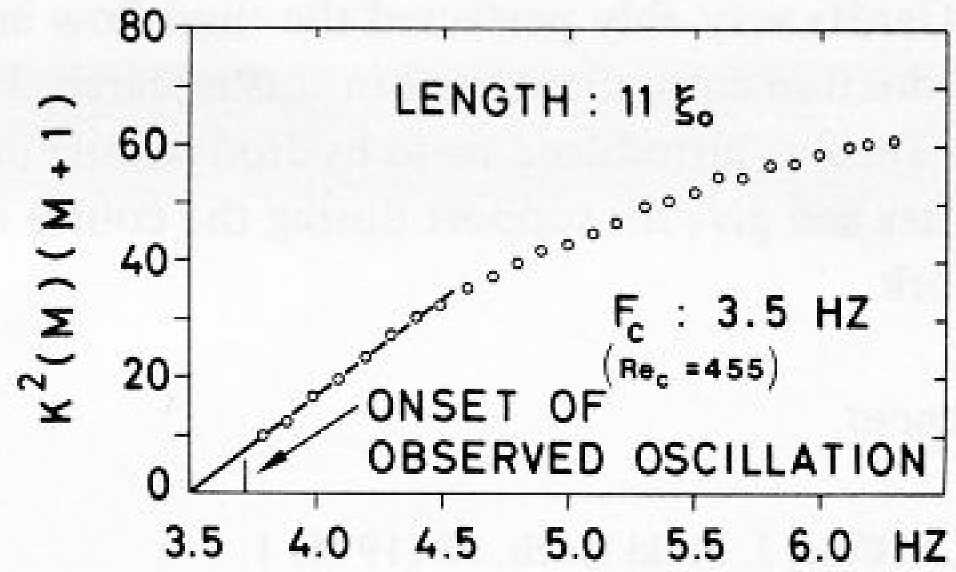

Fig. 5. Plot of $(1 + m)K^2(m)$ (with m obtained from previous fits) versus rotation frequency. Intercept yields critical frequency of infinite geometry.

some approximation, the toroidal Taylor vortices are slightly tilted out of the plane perpendicular to the cylinder axis, and the tilted vortices rotate about the axis. In our measurements this travelling transversal wave manifests itself as an oscillation of the vertical velocity. It has proved convenient to identify the order parameter with the oscillation amplitude of vertical velocity at a spot about 1 mm from the inner cylinder and in a region where there is inward flow associated with the vortices. Since the tilt angle vanishes at top and bottom, so does the order parameter. Fig. 4 shows two measurements of this order parameter versus axial height at two different Reynolds numbers. Fitting the measured data to the function cd of eq. (2c) yields the parameter m and V_m, hence ϵV_0^2 as well. Taking advantage of the periodicity of cd with period $4K(m)$, $K(m)$ being the complete elliptic integral of the first kind, one gets

$$\epsilon = 4(\xi_0/L)^2 K^2(m)(m + 1). \tag{3}$$

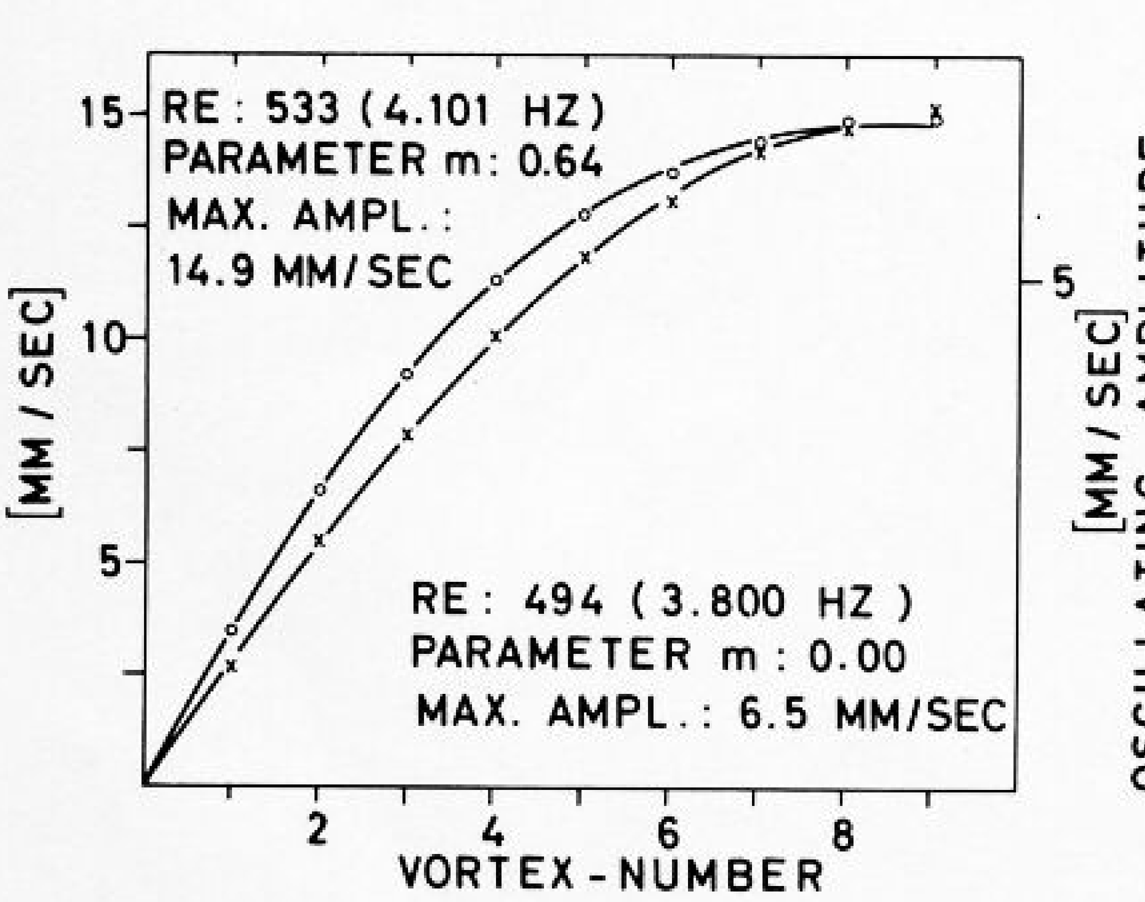

Fig. 4. Measured space-dependent order parameter in the case of the wavy-vortex state. Note that the amplitudes are normalised to show that the shape of the function changes.

Fig. 5 shows a plot of $K^2(m)(m + 1)$ versus rotation rate. The intersection of the resulting straight line with the abscissa gives the critical rotation rate which would apply to an infinitely long cylinder. The evaluation also yields ξ_0 which, as may be expected, is of the order of the gap width.

In summary, our results show that the Landau picture satisfactorily describes both hydrodynamic transitions discussed here, provided the second-order space derivative is included in eq. (1) and the wavelength of the (Taylor vortex) cells is held fixed at a value corresponding to the lowest critical Reynolds number for the onset of stationary Taylor-vortex flow. We present the first experimental proofs for the onset of stationary Taylor vortices as well as the lowest-order time-dependent wavy-vortex mode. In the latter case our work constitutes the first quantitative treatment of this particular phase transition.

In both cases our fitting procedures allow an extrapolation to the critical Reynolds numbers for infinitely long cylinders with a precision virtually limited by uncertainties in the viscosity measurements only. Comparison with Kirchgässner's and Davey's calculations [16,17] yields excellent agreement for the onset of stationary vortices, whereas the formula given by ref. [1] yields a critical Reynolds number for the onset of the wavy mode which is wrong by a factor of 5 under our experimental conditions.

U. Gerdts very ably perfected the fluid flow and data reduction equipment used in this research. E.O. Schulz-DuBois introduced us to hydrodynamic instabilities and give his support during the course of this work.

References

[1] J.A. Cole, J. Fluid Mech. 75 (1976) 1.
[2] E.A. Kuznetsov et al., Institute of Automation and Electrometry, Novosibirsk, USSR, preprint No. 58 (1977).
[3] T.B. Benjamin, Proc. R. Soc. (London) 359 (1978) 1.
[4] T. Mullin and T.B. Benjamin, Nature 288 (1980) 567.
[5] T.B. Benjamin and T. Mullin, Philos. Trans. R. Soc. A, to be published.
[6] P.J. Blennerhassett and P. Hall, Proc. R. Soc. A365 (1979) 191.
[7] D.C. Martin, J. de Phys. 37 (1976) C1-57.
[8] P. Bergé, in: Dynamical critical phenomena and related topics, ed. C.P. Enz (Springer, Berlin, 1979) p. 288.
[9] R.J. Donnelly, Phys. Rev. Lett. 10 (1963) 282.
[10] R.J. Donnelly and K.W. Schwarz, Proc. R. Soc. (London) A283 (1965) 531.
[11] J.P. Gollub and M.H. Freilich, Phys. Fluids 19 (1976) 618;
P.R. Fenstermacher, H.L. Swinney and J.P. Gollub, J. Fluid Mech. 94 (1979) 103.
[12] J. Wesfreid, Y. Yomeau, M. Dubois, C. Normand and P. Bergé, J. de Phys. Lett. 39 (1978) 725.
[13] In this context see L.A. Segel, J. Fluid Mech. 38 (1969) 203;
for a general discussion of eq. (1) see e.g. H. Haken, Synergetics (Springer, Berlin, 1977) p. 186.
[14] M. Abramowitz and I.A. Stegun, Handbook of mathematical functions (Dover Publ., New York, 1972).
[15] R. Vehrenkamp, K. Schätzel, G. Pfister, B.S. Fedders and E.O. Schulz-DuBois, Phys. Scr. 19 (1979) 379.
[16] K. Kirchgässner, Z. Angew. Math. Phys. 12 (1961) 14.
[17] A. Davey, J. Fluid Mech. 31 (1962) 17.
[18] D. Coles, J. Fluid Mech. 21 (1965) 385.
[19] G. Pfister, Proc. 4th Intern. Conf. on Photon correlation techniques in fluid mechanics (Stanford, 1980) p. 27-1.