o-minimality in geometry

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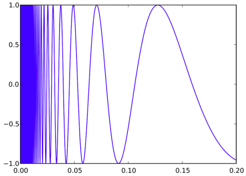
Motivation

Aim of the talk:

- introduce the concept of o-minimality
- sketch the proof of the o-minimal Chow theorem (Peterzil and Starchenko)

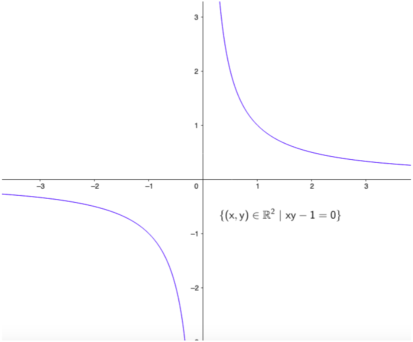
What is o-minimality?

- branch of math is *logic* (here geometric point of view)
- concept of "tame topology": avoid spaces like graph of $x \mapsto \sin(1/x)$



Motivation

- Algebraic sets $\{x \in \mathbb{R}^n \mid P(x) = 0\}$, where $P \in \mathbb{R}[x_1, \ldots, x_n]$ behave better.



- Intersections and unions of algebraic sets are again algebraic.
- Problem: complements and projections. The image of

$$\pi\colon \{(x,y) \in \mathbb{R}^2 \mid xy - 1 = 0\} \to \mathbb{R}, \quad (x,y) \mapsto x$$

is $\mathbb{R}^* = \{x < 0\} \cup \{x > 0\}$.

Definition

A semialgebraic set $A \subset \mathbb{R}^n$ is a subset defined by a boolean combination of polynomial equations and inequalities.

Semialgebraic sets $A \subset \mathbb{R}^n$ form the smallest class $S\mathcal{A}_n$ of subsets of $\mathbb{R}^n$, such that

- $\{x \in \mathbb{R}^n \mid P(x) = 0\} \in S\mathcal{A}_n$ and $\{x \in \mathbb{R}^n \mid P(x) > 0\} \in S\mathcal{A}_n$

  for all $P \in \mathbb{R}[x_1, \ldots, x_n]$
- if $A, B \in S\mathcal{A}_n$, then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A$ belong to $S\mathcal{A}_n$.

Products $V \times W$ of semialgebraic sets $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are again semialgebraic.

Theorem (Tarski-Seidenberg)

Let $A \subset \mathbb{R}^n$ be a semialgebraic set and

$$\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}).$$

Then $p(A) \subset \mathbb{R}^{n-1}$ is semialgebraic.

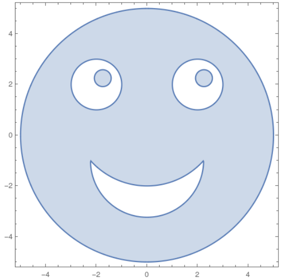
Stability properties of semialgebraic sets

The collection $\mathcal{SA} := \{(\mathcal{SA})_n\}_{n\in\mathbb{N}}$ is stable under

- boolean combinations (intersections, unions and complements),
- products and
- projections.

Moreover each set in $(\mathcal{SA})_1$ is a finite union of intervalls and points.

- Example of a typical semialgebraic set:



Definition

A structure (on $\mathbb{R}$) is a collection $S = (S_n)_{n\in\mathbb{N}}$, such that

1) S_n *is a boolean algebra of subsets of* $\mathbb{R}^n$,
2) *if* $V \in S_n$, *then* $\pi(V) \in S_{n-1}$,
3) *if* $V \in S_n$ *and* $W \in S_m$, *then* $V \times W \in S_{n+m}$,
4) S_n *contains* $\{x \in \mathbb{R}^n \mid P(x) = 0\}$ *for every polynomial* $P \in \mathbb{R}[x_1, \ldots, x_n]$.

A structure S is called o-minimal if

5) *Each set in* S_1 *is a finite union of intervalls and points.*

Example

- *The structure* $S\mathcal{A}$, *where*

$$(S\mathcal{A})_n := \{\text{semialgebraic sets in } \mathbb{R}^n\}$$

  is o-minimal.

- *The smallest structure containing* $\{(x, y) \in \mathbb{R}^2 \mid y = \exp(x)\}$ *is o-minimal (Wilkie).*

Definition/Notation

A subset $V \subset \mathbb{R}^n$ is definable if $V \in S_n$. A function f (or a map) is definable, if its graph is definable.

Proposition

Let $f\colon V \to \mathbb{R}^m$ be definable, then

1) *image and preimage of definable sets under f are definable,*
2) *same for products and compositions of definable functions and maps.*

proof of 1):

- $f(A) = \pi_2(\{(x,y) \in A \times \mathbb{R}^m \mid f(x) = y\})$
- $f^{-1}(B) = \pi_1(\{(x,y) \in V \times \mathbb{R}^m \mid f(x) = y\} \cap (\mathbb{R}^n \times B))$

Example

- *For each $P \in \mathbb{R}[x_1, \ldots, x_n]$ the set $\{x \in \mathbb{R}^n \mid P(x) > 0\}$ is definable:*

$$\{x \in \mathbb{R}^n \mid P(x) > 0\} = \pi_x(\{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1}^2 P(x) = 1\}.$$

- *The $\sin$-function is not definable in any o-minimal structure, since $\sin^{-1}(0)$ is infinite and discrete.*

Conclusion

Every structure S contains $S\mathcal{A}$. (Semialgebraic sets form the smallest structure).

Proposition

Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and definable in some o-minimal structure, then $f \in \mathbb{C}[z]$.

proof:

- otherwise $g(z) := f(1/z)$ has an essential singularity at 0.
- Let W^* be a punctured nbhd of the origin

$$\implies \quad g(W^*) = \mathbb{C} \quad \text{or} \quad g(W^*) = \mathbb{C} \setminus \{p\}.$$

  Moreover each value is taken infinitely many times (Picard's great theorem)

$$\implies \text{we may assume that } f \text{ has infinitely many zeros}$$

- zeros of $f(z)$ cannot accumulate $\implies$ $f^{-1}(0)$ is not definable (a contradiction).

Definition

A first order formula in S is constructed by the following rules:

1) $A \in S_n \implies x \in A$ *is a formula,*
2) $P \in \mathbb{R}[x_1, \ldots, x_n]$ *then* $P = 0$, $P > 0$ *and* $P < 0$ *are formulas,*
3) ϕ *and* ψ *are formulas* $\implies \psi \vee \phi$, $\psi \wedge \phi$ *and* $\neg\phi$ *are formulas,*
4) *if* $\phi(x, y)$ *is a formula and A is definable, then*

$$\exists x \in A \quad s.t. \quad \phi(x, y) \qquad \textit{and} \qquad \forall x \in A \quad \phi(x, y)$$

are formulas.

Proposition

If ϕ *is a first order formula, then* $\{x \in \mathbb{R}^n \mid \phi(x)\}$ *is definable.*

proof: Rules 1–3 produce definable sets. Rule 4: suppose

$$B = \{(x, y) \in \mathbb{R}^{n+p} \mid \phi(x, y)\}$$

is definable, then

- $\{y \in \mathbb{R}^p \mid \exists x \in A \; s.t. \; \phi(x, y)\} = \pi_y\big(B \cap (A \times \mathbb{R}^p)\big)$ and
- $\{y \in \mathbb{R}^p \mid \forall x \in A \;\; \phi(x, y)\} = \mathbb{R}^p \setminus \pi_y\big((A \times \mathbb{R}^p) \cap (\mathbb{R}^{n+p} \setminus B)\big)$ are definable.

Example

1) *Let $A \in S_n$ be definable, then*

$$\overline{A} = \{x \in \mathbb{R}^n \mid \forall \epsilon \in \mathbb{R},\ \epsilon > 0,\ \exists y \in A\ \ s.t.\ \sum_{i=1}^{n}(x_i - y_i)^2 < \epsilon^2\}.$$

is definable and therefore also $fr(A) := \overline{A} \setminus A$.

2) *The derivative of a differentiable and definable function is definable.*

Theorem (Peterzil, Starchenko)

Let $Y \subset \mathbb{C}^n$ be a closed, analytic and definable in some o-minimal structure, Y is complex algebraic.

Analytic and algebraic subsets

- $Y \subset \mathbb{C}^n$ is analytic $\iff$ for all $y \in \mathbb{C}^n$ there exists $U = U(y) \subset \mathbb{C}^n$ open and $f_1, \ldots, f_k \in \mathcal{O}(U)$ s.t. $U \cap Y = \{x \in U \mid f_1(x) = \ldots = f_k(x) = 0\}$.
- $Y \subset \mathbb{C}^n$ is complex algebraic $\iff$ there exists $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$ s.t.
$$Y = \{x \in \mathbb{C}^n \mid f_1(x) = \ldots = f_k(x) = 0\}.$$

Proof of o-minimal Chow

- We prove the theorem in several steps:

Step 1

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be holomorphic and definable, then $f \in \mathbb{C}[x_1, \ldots, x_n]$.

Step 2 ("Noether normalization")

There exists a finite, proper and definable map $\pi\colon\ Y \to \mathbb{C}^d$, where $d := \dim_{\mathbb{C}}(Y)$.

Step 3

The branch locus $\mathcal{B}$ of $\pi\colon\ Y \to \mathbb{C}^d$ is definable.

Step 4

Use the associated unramified cover $\pi\colon\ Y \setminus \mathcal{R} \to \mathbb{C}^d \setminus \mathcal{B}$ to derive polynomial equations for Y.

Proof of o-minimal Chow

Step 2 ("Noether normalization")

There exists a finite, proper and definable map $\pi\colon Y \to \mathbb{C}^d$, where $d := \dim_{\mathbb{C}}(Y)$.

proof:

- We embed Y in projective space

$$Y \subset \mathbb{C}^n \to \mathbb{P}^n, \qquad x \mapsto (1 : x)$$

- the frontier $fr(Y) = \overline{Y} \setminus Y \subset H := \{x_0 = 0\} \simeq \mathbb{P}^{n-1}$ is definable
- fact from o-minimality:

$$\dim_{\mathbb{R}} \big(fr(Y)\big) < \dim_{\mathbb{R}}(Y) = 2d \quad \Longrightarrow \quad fr(Y) \neq H$$

- pick $p \in H \setminus \overline{Y}$ and consider the projection

$$\pi\colon \overline{Y} \to \mathbb{P}^{n-1}, \quad (x_0 : x) \to (x_0 : L_1(x) : \cdots : L_{n-1}(x))$$

- the restriction $\pi\colon Y \to \mathbb{C}^{n-1}$ is proper, holomorphic, definable and finite
- the image $\pi(Y) \subset \mathbb{C}^{n-1}$ is closed, analytic (Remmert) and definable (then we repeat)

Step 3

The branch locus $\mathcal{B}$ of $\pi\colon Y \to \mathbb{C}^d$ is definable.

Let $m := \deg(\pi)$ be the degree of π, then

$$\mathcal{B} = \{x \in \mathbb{C}^d \mid \#\pi^{-1}(x) < m\}.$$

proof:

- we may assume that $Y \subset \mathbb{C}^d \times \mathbb{C}^{n-d}$ and $\pi(x, y) = x$
- consider the definable set

$$M_m := \{(x_1, y_1, \ldots, x_m, y_m) \in Y^m \mid x_1 = \ldots = x_m\} \setminus \bigcup_{i<j} \{y_i = y_j\}$$

- The complement $\mathbb{C}^d \setminus \mathcal{B}$ is definable, because it is the image of the projection

$$\pi\colon M_m \to \mathbb{C}^d, \qquad (x_1, y_1, \ldots, x_m, y_m) \mapsto x_1$$

Proof of o-minimal Chow

Step 4

Use the associated unramified cover $\pi\colon Y \setminus \mathcal{R} \to \mathbb{C}^d \setminus \mathcal{B}$ to derive polynomial equations for Y.

proof:

- we write $Y_0 := Y \setminus \mathcal{R}$ and $V := \mathbb{C}^d \setminus \mathcal{B}$.
- for all $p \in V$ there exists $W = W(p) \subset V$ open such that

$$\pi^{-1}(W) = \bigsqcup_{i=1}^{m} U_i \qquad \text{and} \qquad \pi_{|U_i}\colon U_i \to W \qquad \text{biholomorphic}$$

- the inverse maps are of the form

$$(\pi_{|U_i})^{-1}\colon W \to U_i, \qquad x \mapsto (x, \phi_i(x)) \in \mathbb{C}^d \times \mathbb{C}^{n-d}$$

- **for simplicity:** $d = n - 1$ (we need only one equation).

$$\begin{aligned}\pi^{-1}(W) &= \{(x,y) \in W \times \mathbb{C} \mid \prod (y - \phi_i(x)) = 0\} \\ &= \{(x,y) \in W \times \mathbb{C} \mid y^m - s_1(x)y^{m-1} + \ldots + (-1)^m s_m(x) = 0\}\end{aligned}$$

Proof of o-minimal Chow

- the functions $s_i := \sum_{k_1 < \cdots < k_i} \phi_{k_1} \cdots \phi_{k_i}$ glue to functions on $V = \mathbb{C}^d \setminus \mathcal{B}$
- by properness s_i extend to an entire function $\widehat{s_i} \colon \mathbb{C}^d \to \mathbb{C}$
- **claim:** $\widehat{s_i}$ is definable and therefore a polynomial (Step 1)

$$\implies \quad Y = \overline{Y_0} = \{(x, y) \mid y^m - \widehat{s_1}(x) y^{m-1} + \ldots + (-1)^m \widehat{s_m}(x) = 0\}$$

<u>proof of claim:</u>

- recall the definable set

$$M_m := \{(x_1, y_1, \ldots, x_m, y_m) \in Y^m \mid x_1 = \ldots = x_m\} \setminus \bigcup_{i<j} \{y_i = y_j\}$$

- $\Gamma(s_i)$ is the image of the definable map

$$\psi_i \colon M_m \to V \times \mathbb{C}, \quad (x_1, y_1, \ldots, x_m, y_m) \mapsto \left(x_1, \sum_{k_1 < \cdots < k_i} y_{k_1} \cdots y_{k_i}\right)$$

$$\implies \quad \widehat{s_i} \text{ is also definable}$$

Thank you for your attention