On the degree of the canonical map of a product quotient surface

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Motivation

- For a smooth complex algebraic surface S of general type we can consider its canonical map

$$\varphi_{K_S}\colon S \dashrightarrow \mathbb{P}^{p_g-1}, \qquad p_g(S) := h^0(\mathcal{O}(K_S)) = h^0(\Omega_S^2) \quad (\geq 1)$$

- If φ_{K_S} is generically finite then automatically $p_g(S) \geq 3$
- Beauville '79 remarked an upper bound for $deg(\varphi_{K_S})$:

$$9\chi(\mathcal{O}_S) \geq K_S^2 \geq deg(\varphi_{K_S}) \cdot deg(\varphi_{K_S}(S)) \geq deg(\varphi_{K_S}) \cdot (p_g(S) - 2)$$

$\Longrightarrow$ resolving for the degree yields:

Beauvilles upper bound

$$deg(\varphi_{K_S}) \leq \frac{9\chi(\mathcal{O}_S)}{p_g(S)-2} = \frac{9\big(1 - q(S) + p_g(S)\big)}{p_g(S)-2} = 9 + \frac{27 - 9q(S)}{p_g(S)-2} \leq 36.$$

($q(S) := h^0(\Omega_S^1)$ is the irregularity of S)

Motivation

The inequality

- $9\chi(\mathcal{O}_S) \geq K_S^2 \quad (BMY)$ is sharp iff S is a ball quotient
- $K_S^2 \geq deg(\varphi_{K_S}) \cdot deg(\varphi_{K_S}(S))$ is sharp iff $|K_S|$ has no basepoints (b.p.f)

Remark

To realize the upper bound $deg(\varphi_{K_S}) = 36$ we need:

- *S to be a ball quotient,*
- *the canonical system $|K_S|$ b.p.f,*
- *$p_g(S) = 3$ and $q(S) = 0$.*

- In 78' Persson provided an example with $deg(\varphi_{K_S}) = 16$, the highest known degree until 2015
- In 2015 Yeung claimed to have an example with $deg(\varphi_{K_S}) = 36$, but the proof seems to have a gap.

Motivation

The main result (Pignatelli, Rito, -)

A realization of $deg(\varphi_{K_S}) = 32$.

- Ball quotients are very hard to handle

$\implies$ surfaces with $K_S^2 = 8\chi(\mathcal{O}_S)$ are simpler

- For these surfaces the inequality becomes

$$deg(\varphi_{K_S}) \leq \frac{8\chi(\mathcal{O}_S)}{p_g(S) - 2} = 8 + \frac{24 - 8q(S)}{p_g(S) - 2} \leq 32$$

Remark

To realize the upper bound $deg(\varphi_{K_S}) = 32$ we need:

- *the canonical system $|K_S|$ b.p.f,*
- *$p_g(S) = 3$ and $q(S) = 0$.*

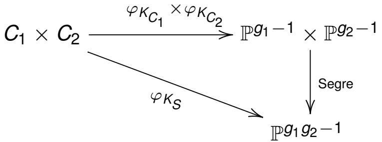
- A typical example of a surface (of general type) with

$$K_S^2 = 8\chi(\mathcal{O}_S)$$

is a product of two smooth projective curves C_i of genus $g_i \geq 2$:

$$8\chi(\mathcal{O}_S) = 8\chi(\mathcal{O}_{C_1}) \cdot \chi(\mathcal{O}_{C_2}) = 8(1 - g_1)(1 - g_2) = K_S^2$$

- The canonical map of $S = C_1 \times C_2$ factors:



Classical Theorem

- $\varphi_{K_{C_i}}$ is an embedding $\iff$ C_i is not hyperelliptic
- If C_i is hyperelliptic then $\varphi_{K_{C_i}}$ is of degree 2

$\Longrightarrow$ too small to be interesting

The idea:

Generalize the situation by taking quotients of a product of curves.

Definition

A complex projective surface S is said to be isogenous to a product if S is a quotient

$$S = (C_1 \times C_2)/G,$$

where the C_i's are smooth curves of genus at least two, and G is a finite group acting freely on $C_1 \times C_2$.

Motivation

Remark

- A surface isogenous to a product is smooth, minimal, of general type i.e. $\kappa(S) = 2$ and K_S is ample.
- Simple formulas for the invariants:

$$\chi(\mathcal{O}_S) = \frac{(g_1 - 1)(g_2 - 1)}{|G|}, \qquad K_S^2 = 8\chi(\mathcal{O}_S) \qquad \text{and} \qquad e(S) = 4\chi(\mathcal{O}_S).$$

The initial example is due to Beauville:

- He considered a free $(\mathbb{Z}/5)^2$ action on two copies of the Fermat quintic

$$C := \{x^5 + y^5 + z^5 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$$

- Since $g(C) = (5-1)(5-2)/2 = 6$ the quotient surface

$$S = (C \times C)/(\mathbb{Z}/5)^2$$

has $\chi(\mathcal{O}_S) = 1$.

Motivation

- For a surface S of general type it holds $\chi(\mathcal{O}_S) \geq 1$.
- There is a complete classification of all surfaces isogenous to a product in the boundary case $\chi(\mathcal{O}_S) = 1$ due to Bauer, Catanese, Grunewald, Carnovale, Polizzi, Pennegini, Zucconi.
- In principle the methods to classify these surfaces work also for other values for $\chi(\mathcal{O}_S)$. But the classification becomes much harder as $\chi(\mathcal{O}_S)$ increases.

The problems

I) We have to construct examples of surfaces S isogenous to a product for fixed invariants p_g and q, if possible a full classification

$\Longrightarrow$ in view of Beauvilles inequality the interesting values are

$$p_g(S) = 3, 4 \qquad \text{and} \qquad q(S) = 0.$$

II) Once we have the examples, we need to analyse their canonical system: basepoints and the image of φ_{K_S} in $\mathbb{P}^{p_g - 1}$.

Working assumptions

- For simplicity we assume that the action on $C_1 \times C_2$ is diagonal

$$g(x,y) = (g \cdot x, g \cdot y) \qquad \forall \qquad g \in G.$$

- According to Catanese there is a unique realisation of S such that G acts faithfully on each curve.

Proposition

Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product, then it holds

$$q(S) = g(C_1/G) + g(C_2/G).$$

- Since we are interested in the case $q(S) = 0$, we also assume $C_i/G \simeq \mathbb{P}^1$.

Observation:

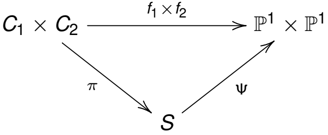
Since we act on the product $C_1 \times C_2$ via the diagonal subgroup $\Delta \leq G \times G$, we obtain

- (branched) Galois covers $f_i \colon C_i \to C_i/G = \mathbb{P}^1$ of the projective line and
- a branched cover

$$\psi \colon S \to (C_1 \times C_2)/(G \times G) = C_1/G \times C_2/G \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

Working assumptions

- The quotient map $\pi\colon C_1 \times C_2 \to S$ and the covers ψ and f_i fit in a commutative triangle



$\Longrightarrow$ The cover ψ is particularly useful when it is Galois. This is the case iff Δ is normal in $G \times G$ i.e. G is abelian.

- Our final assumption is: G is abelian

- This allows us to apply the powerful theory of **abelian covers** to $\psi\colon S \to \mathbb{P}^1 \times \mathbb{P}^1$, which has Galois group

$$(G \times G)/\Delta \simeq G, \qquad (a, b) \mapsto ab^{-1}.$$

- It also makes a classification for a fixed p_g more feasible.

Monodromy and Riemann's existence theorem

Remark

- In Beauvilles example we had an explicit description in terms of equations
- In general it is hard to work with equations, so we aim for a more abstract description

The Galois covers $f_i \colon C_i \to C_i/G = \mathbb{P}^1$ are described by:

Riemann's existence theorem

Given

- a finite set $\mathcal{B} \subset \mathbb{P}^1$ and
- a surjective group homomorphism $\eta \colon \pi_1(\mathbb{P}^1 \setminus \mathcal{B}) \to G$ to a finite group G

then, there exists up to isomorphism a unique compact Riemann surface C and a unique inclusion $G \subset Aut(C)$ such that

- $C/G \simeq \mathbb{P}^1$,
- the branch locus of $C \to \mathbb{P}^1$ is contained in $\mathcal{B}$
- the monodromy of $C \setminus \mathcal{R} \to \mathbb{P}^1 \setminus \mathcal{B}$ is η.

- Recall that $\pi_1(\mathbb{P}^1 \setminus \mathcal{B})$ has a presentation of the form $\langle \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^r \gamma_i = 1 \rangle$
- The images $a_i = \eta(\gamma_i)$ generate the group G and fulfill the relation

$$\prod_{i=1}^{r} a_i = 1$$

- Let m_i be the order of a_i, then Hurwitz' formula holds:

$$2g(C) - 2 = |G| \Big(-2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \Big)$$

Definition

Let $m_1, \ldots, m_r \geq 2$ be integers, a **spherical system of generators** for a finite group G of type $[m_1, \ldots, m_r]$ is an r-tuple

$$(a_1, \ldots, a_r)$$

of elements generating G, such that

$$\prod_{i=1}^{r} a_i = 1 \qquad \text{and} \qquad \mathrm{ord}(a_i) = m_i.$$

Monodromy and Riemann's existence theorem

- We define the *stabilizer set* of a generating vector $V = (a_1, \ldots, a_r)$ as

$$\Sigma_V := \bigcup_{g \in G} \bigcup_{i \in \mathbb{Z}} \bigcup_{j=1}^{r} \left\{ g a_j^i g^{-1} \right\}$$

- To a surface isogenous to a product

$$S = (C_1 \times C_2)/G,$$

  we can attach two generating vectors V_1 and V_2.

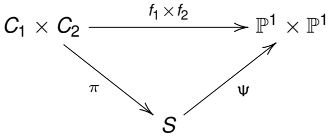
- The freeness of the G-action on the product $C_1 \times C_2$ is reflected by the condition

$$\Sigma_{V_1} \cap \Sigma_{V_2} = \{1_G\}.$$

- Conversely a triple (G, V_1, V_2) consisting of a finite group and two generating vectors whose stabilizer sets intersect trivially is realized by a surface isogenous to a product.

The abelian cover ψ

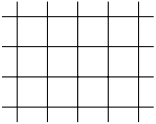
- Next we study the abelian $G \simeq (G \times G)/\Delta$ cover ψ in the diagram



- Since π is by assumption unramified, $f_1 \times f_2$ and ψ have the same branch locus:

$$\mathcal{B} = \mathbb{P}^1 \times \mathcal{B}_2 \quad \cup \quad \mathcal{B}_1 \times \mathbb{P}^1,$$

where $\mathcal{B}_1 = \{p_1, \ldots, p_s\}$ is the branch locus of f_1 and $\mathcal{B}_2 = \{q_1, \ldots, q_l\}$ the branch locus of f_2:



$$B_i^{ver} = p_i \times \mathbb{P}^1 \qquad \text{and} \qquad B_j^{hor} = \mathbb{P}^1 \times q_j$$

Computing a basis of $H^0(S, K_S)$

- The G action on $\psi_*\mathcal{O}_S$ induce a splitting in eigensheaves

$$\psi_*\mathcal{O}_S = \bigoplus_{\chi \in Irr(G)} (\psi_*\mathcal{O}_S)^\chi$$

that are locally free of rank 1 $\implies$ $(\psi_*\mathcal{O}_S)^\chi = \mathcal{O}(L_\chi^{-1})$ for some line bundle

- According to a result of Catanese, the line bundles L_χ can be determined from the monodromy.

Theorem (special case of Catanese, Liedtke)

Let $S = (C_1 \times C_2)/G$ be a regular surface isogenous to a product, where G is abelian. Let

$$V_1 = (a_1, \ldots, a_s) \qquad \text{and} \qquad V_2 = (b_1, \ldots, b_l)$$

be associated generating vectors, $m_i = ord(a_i)$ and $n_j = ord(b_j)$.
Let R_i^{vert} and R_j^{hor} be the reduced inverse images of B_i^{vert} and B_j^{hor} under ψ, then

$$H^0(S, K_S) = \bigoplus_{\chi \in Irr(G)} s_\chi \cdot H^0(\mathcal{O}(-2,-2) \otimes L_\chi),$$

where

$$(s_\chi) = \sum_{i=1}^{s} \left(m_i - 1 - \frac{m_i \chi(a_i)}{d_\chi}\right) R_i^{vert} + \sum_{j=1}^{l} \left(n_j - 1 - \frac{n_j \chi(b_j^{-1})}{d_\chi}\right) R_j^{hor}, \qquad d_\chi := ord(\chi).$$

Computing a basis of $H^0(S, K_S)$

- only vertical and horizontal divisors can intersect
- no basepoints can come from $H^0\left(\mathcal{O}(-2,-2) \otimes L_\chi\right)$

Remark

The canonical divisor K_S of a surface isogenous to a product is ample

$$\Longrightarrow \varphi_{K_S}(S) \text{ is a surface if } |K_S| \text{ is b.p.f.}$$

- The first step towards a classification for a fixed value of p_g is a bound on the group order.

Proposition

Let $S = (C_1 \times C_2)/G$ be a regular surface isogenous to a product, where G is abelian, then

$$n := |G| \leq 8(p_g + 2) + 8\sqrt{(p_g + 2)^2 - 1}$$

In particular

$$n \leq \begin{cases} 79 & \text{for } p_g(S) = 3 \\ 95 & \text{for } p_g(S) = 4 \end{cases}$$

Proof.

Since G is abelian, we have $n \leq 4(g_i + 1)$. This implies

$$n \cdot (1 + p_g) = (g_1 - 1)(g_2 - 1) \geq \left(\frac{n}{4} - 2\right)^2,$$

which is equivalent to

$$0 \geq n^2 - 16(p_g + 2)n + 64.$$

□

- Let $S = (C_1 \times C_2)/G$ be a regular surface isogenous to a product, where G is abelian, then G cannot be cyclic (Bauer and Catanese).

- Let V_i be the generating vectors. Then the types

$$T_1 = [m_1, \dots, m_s] \qquad \text{and} \qquad T_2 = [n_1, \dots, n_l]$$

fulfill additional constraints:

- n_i divides $g(C_1) - 1$ and m_j divides $g(C_2) - 1$
- $lcm\{m_1, \dots, \widehat{m}_i, \dots, m_s\} = lcm\{m_1, \dots, m_s\}$ and the same for $T_2 = [n_1, \dots, n_l]$.
- Hurwitz' formula

$$2g(C_1) - 2 = |G|\Big(-2 + \sum_{i=1}^{s} \frac{m_i - 1}{m_i}\Big)$$

and also for T_2.
- other constraints ...

Remark

For a fixed value of p_g the classification problem becomes finite.

Input: a positive integer $p_g \geq 3$

1st Step: here we determine the set of triples

$$(n, T_1, T_2), \quad \text{where the types} \quad T_1 = [m_1, \ldots m_s] \quad \text{and} \quad T_2 = [n_1, \ldots, n_l]$$

satisfy the constraints from above.

2nd Step:

- For each triple (n, T_1, T_2) found in the 1st step we run through the abelian groups G of order n and determine all triples of the form (G, V_1, V_2) where V_i is a generating vector for G of type T_i.
- We check the freeness condition $\Sigma_{V_1} \cap \Sigma_{V_2} = \{1_G\}$

  $\Longrightarrow$ we obtain a surface S isogenous to a product
- for each surface S we check if $|K_S|$ is b.p.f.

Output: a complete list of all regular surfaces isogenous to a product with geometric genus p_g, obtained by the action of an abelian group such that $|K_S|$ is b.p.f.

Automatically $deg(\varphi_{K_S}) \leq 8 + \frac{24}{p_g - 2}$ with equality iff $deg(\varphi_{K_S}(S)) = p_g - 2$.

The results:

We run a MAGMA implementation of the algorithm for the input value $p_g(S) = 3$ we obtain:

Theorem (Pignatelli, Rito, -)

There are precisely two families of regular surfaces S isogenous to a product with $p_g(S) = 3$ obtained by the action of an abelian group such that $|K_S|$ is b.p.f.

$$\Longrightarrow \quad deg(\varphi_{K_S}) = 32.$$

The 1st example in detail:

- The group is $G = (\mathbb{Z}/2)^4$
- The generating vectors are:

$$V_1 = (e_{124}, e_1, e_{1234}, e_{23}, e_{14}, e_{24}), \qquad V_2 = (e_{34}, e_2, e_3, e_{13}, e_{123}, e_4),$$

where $e_{ijk} := e_i + e_j + e_k$ is the sum of the unit vectors.

$\Longrightarrow$ There are six vertical and six horizontal branch divisors of

$$\psi \colon S \to \mathbb{P}^1 \times \mathbb{P}^1.$$

The results:

- Let $\chi_1, \ldots, \chi_{16}$ be the irreducible characters of $(\mathbb{Z}/2)^4$ (ordered as in MAGMA's character table), then:

$$h^0(\mathcal{O}(-2,-2) \otimes L_{\chi_i}) = \begin{cases} 1 & \text{if } i = 8, 12, 13 \\ 0 & \text{else} \end{cases}$$

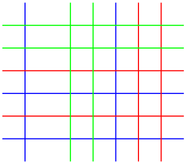
- A basis for $H^0(S, K_S)$ is given by $(s_{\chi_8}, s_{\chi_{12}}, s_{\chi_{13}})$.

$$(s_{\chi_8}) = R_1^{vert} + R_4^{vert} + R_4^{hor} + R_6^{hor} \qquad \text{(blue)}$$

$$(s_{\chi_1 1}) = R_5^{vert} + R_6^{vert} + R_3^{hor} + R_5^{hor} \qquad \text{(red)}$$

$$(s_{\chi_1 3}) = R_2^{vert} + R_3^{vert} + R_1^{hor} + R_2^{hor} \qquad \text{(green)}$$

The picture in the branch locus:



What happens for $p_g = 4$?

- Here the maximal possible degree of φ_{K_S} is 20.

  To realize this bound we need an example with $|K_S|$ b.p.f. such that the image of φ_{K_S} is a quadric.

- Unfortunately when we run the program for $p_g = 4$, we find that all examples have basepoints.

Generalizations:

- We can drop the freeness assumption and allow mild singularities i.e. canonical singularities.

- The stabilizer of a point $(x, y) \in C_1 \times C_2$ is the intersection

$$Stab(x, y) = Stab(x) \cap Stab(y)$$

of cyclic groups and therefore cyclic.

$\implies$ The quotient S has at most finitely many **isolated cyclic quotient singularities**.

- A germ of a canonical cyclic quotient singularity (in dimension two) is represented by $\mathbb{C}^2$ modulo

$$\left\langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{n-1} \end{pmatrix} \right\rangle \leq SL(2, \mathbb{C}), \qquad \text{where} \qquad \epsilon := \exp\left(\frac{2\pi i}{n}\right)$$

and denoted by A_{n-1}.

Generalizations:

- Under this assumption the canonical divisor K_S of $S = (C_1 \times C_2)/G$ is Cartier and S has a crepant resolution:

$$\rho\colon \widehat{S} \to S, \quad \text{where } \widehat{S} \text{ is smooth, } \rho \text{ is proper and birational and} \quad \rho^* K_S = K_{\widehat{S}}$$

- The invariants $\chi(\mathcal{O}_S)$ and K_S^2 are related by:

$$8\chi(\mathcal{O}_S) \geq 8\chi(\mathcal{O}_S) - \frac{2}{3} \sum_{x \in Sing(S)} \frac{n_x^2 - 1}{n_x} = K_S^2$$

$\Longrightarrow$ there might be still useful examples with few singularities

- We extend the algorithm to the canonical case and find an example with $p_g(S) = 4$ and b.p.f. canonical system.

- The example has nine singularities of type A_2 and maps to the quadratic cone

$$uv = w^2.$$

- $K_S^2 = 24 \quad \Longrightarrow \quad deg(\varphi_K) = 12$

Generalizations:

- Another way is to try with irregular surfaces (isogenous to a product) e.g

$$\psi\colon S \to \mathbb{P}^1 \times E, \qquad \text{where } E \text{ is an elliptic curve i.e. } q(S) = 1$$

Theorem (Pignatelli, Rito)

There is a surface S isogenous to a product with $p_g(S) = 3$ and $q(S) = 1$ such that $|K_S|$is basepoint free

$$\Longrightarrow \quad deg(\varphi_K) = 24.$$

- Pignatelli and Rito found their example via a hand computation.
- Up to now no systematic (computer-) search has been performed for irregular surfaces isogenous to a product.

Thank you for your attention