Varieties Isogenous to a Product

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Outline

1. What are varieties isogenous to a product?
2. The surface case $n = 2$
3. The threefold case $n = 3$

Definition

A complex projective variety X is said to be *isogenous to a product* if X is a quotient

$$X = (C_1 \times ... \times C_n)/G,$$

where the C_i's are curves of genus at least two, and G is a finite group acting *freely* on $C_1 \times ... \times C_n$.

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Remark: For the rest of the talk we consider the *unmixed* case where the action of G on the Product $C_1 \times ... \times C_n$ is diagonal i.e. $G = G \cap (Aut(C_1) \times ... \times Aut(C_n))$.
We assume furthermore that G acts faithfully on each curve.

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- interesting relations with group theory and computer algebra

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Classifications for fixed invariants $p_g(X) = h^0(X, \Omega_X^2)$, $q(X) = h^0(X, \Omega_X^1)$:

- $p_g = 0$, $q = 0$ Bauer, Catanese, Grunewald [BCG08],
- $p_g = 1$, $q = 1$ Carnovale, Polizzi [CP09]
- $p_g = 2$, $q = 2$ Penegini [Pe10].

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- $p_g = 1$, $q = 1$ Carnovale, Polizzi [CP09]
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Aim: Classification for the invariants $p_g = 1$ and $q = 0$.

- Start with expressing numerical invariants of $S = (C_1 \times C_2)/G$ in terms of $g(C_1)$ and $g(C_2)$:

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In our case $p_g = 1,\ q = 0 \implies \chi(\mathcal{O}_S) = 2$. It follows

$$|G| = \frac{(g(C_1) - 1)(g(C_2) - 1)}{2}, \quad K_S^2 = 16 \quad \text{and} \quad e(S) = 8.$$

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Since $q(S) = h^0(C_1 \times C_2, \Omega^1_{C_1 \times C_2})^G$ and we have fixed $q(S) = 0$

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- Study G-Covers of $\mathbb{P}^1_{\mathbb{C}}$ in greater detail.

Definition

Let G be a finite group, $2 \leq m_1 \leq ... \leq m_r$ integers.
A spherical system of generators of G (ssg) of type $[m_1, ..., m_r]$ is a r-tuple $A = (g_1, ..., g_r)$ of elements of G s.t.

- $G =< A >$,
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$\implies$ unramified G-cover $C^* \to \mathbb{P}^1_{\mathbb{C}} - \{P_1, \dots, P_r\}$.

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Theorem (Riemann's existence theorem)

A finite group G acts as a group of automorphisms of some compact Riemann surface C s.t. $C/G \simeq \mathbb{P}^1_{\mathbb{C}}$ *iff*

- $\exists$ *ssg of type* $[m_1, ..., m_r]$,
- *Hurwitz' formula holds:*

$$2g(C) - 2 = |G|(-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})).$$

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- Geometrically disjoint means that G acts without fixed points on $C_1 \times C_2$.

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- Vice versa, the data above determine a surface isogenous to a product.

Theorem (finiteness of classification)

There are only finitely many groups G, acting fixed point free and diagonally on a product $C_1 \times C_2$, ($g(C_i) \geq 2$), such that $p_g(S) = 1$ and $q(S) = 0$, where

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$$\implies 16 = K_S^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G|} \geq \frac{2|G|^2}{42^2|G|}$$

therefore $|G| \leq 14112$.

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We can show, as in the last theorem:

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Theorem (-)

There are exactly 49 families of surfaces isogenous to a product of unmixed type with $p_g = 1$ and $q = 0$.

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  Similar for 1-forms and 3-forms.
- We need to understand the G-module structure of $H^0(C, \Omega^1_C)$, where $C \to \mathbb{P}^1_{\mathbb{C}}$ is a G-Cover.

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$ord(g_i) = m_i \implies$ every eigenvalue of $\varrho_j(g_i)$ is of the form $\xi_{m_i}^\alpha = exp(\frac{2\pi\sqrt{-1}\alpha}{m_i})$, $1 \leq \alpha \leq m_i$.

- Let $A = (g_1, ..., g_r)$ be a corresponding ssg of type $[m_1, ..., m_r]$.
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- Define $N_{i,\alpha} := \#$ eigenvalues of $\varrho_j(g_i)$ equal to $\xi_{m_i}^\alpha$.
- Formula of Chevalley-Weil:

$$n_j = -d_j + \sum_{i=1}^{r} \sum_{\alpha=1}^{m_i} N_{i,\alpha}(1 - \frac{\alpha}{m_i}) + \sigma,$$

  where $d_j = dim(V_j)$ and $\sigma = 1$ if ϱ_j is trivial else $\sigma = 0$.

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- The character χ of $H^0(Y, \Omega^2_Y)$ is:

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- $q_2(X) = h^0(X, \Omega^2_X) = h^0(Y, \Omega^2_Y)^G =< \chi, \chi_{triv} >$.

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- We have to drop the assumption that G acts freely on $Y = C_1 \times C_2 \times C_3$ and allow *singularities*.

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$$\begin{pmatrix} \exp(\frac{2\pi i a}{n}) & 0 & 0 \\ 0 & \exp(\frac{2\pi i b}{n}) & 0 \\ 0 & 0 & \exp(\frac{2\pi i c}{n}) \end{pmatrix}$$

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  where $1 \leq a, b, c \leq n$. We write $\frac{1}{n}(a, b, c)$.
- Singularity is *isolated* iff $gcd(a, n) = gcd(b, n) = gcd(c, n) = 1$.

- We want to allow *canonical (or terminal) singularities only*. We can see this from the numbers n, a, b, c.

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Theorem ([Reid87])

A cyclic quotient singularity of type $\frac{1}{n}(a, b, c)$ is terminal (or canonical) iff

$$\alpha_k := \frac{1}{n}(\overline{ka} + \overline{kb} + \overline{kc}) > 1 \quad \textit{for} \quad 1 \leq k \leq n-1,$$

(respectively ≥ 1). Here $\overline{d}$ denotes smallest residue mod n.

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(respectively ≥ 1*). Here* $\overline{d}$ *denotes smallest residue mod n.*

- Consider a resolution of singularities:

$$\widetilde{X} \to Y/G, \;\; \text{where} \;\; Y = C_1 \times C_2 \times C_3.$$

By [F71] we have $h^0(\widetilde{X}, \Omega^i_{\widetilde{X}}) = h^0(Y, \Omega^i_Y)^G$

- The *canonical volume* is:

$$K_X^3 = \frac{48(g(C_1) - 1)(g(C_2) - 1)(g(C_3) - 1)}{|G|} \in \mathbb{Q}.$$

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Theorem

Let $c > 0$, and $K_X^3 \leq c$. If $q_1(X) = 0$, then we have the following bounds:

i) $|G| \leq \lfloor 42\sqrt{c \cdot 7} \rfloor$,

ii) $l_i \leq \lfloor \frac{c}{12} + 4 \rfloor$,
where l_i is the number of branch points of $f_i : C_i \to \mathbb{P}^1_{\mathbb{C}}$.

- For $c = 16$ we get $|G| \leq 444$ and $l_i \leq 5 \implies$ a computer search is possible. We find:

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    - eight different groups,
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  - eight different groups,
  - largest group order $|G| = 192$,
  - largest $q_2 = 6$,
  - smallest $K_X^3 = 4$.
- Remark: In the smooth case we have the following equality:

$$K_X^3 = -48\chi(\mathcal{O}_X).$$

If we fix $\chi(\mathcal{O}_X)$ and $q_1(X) = 0$, then we have

$$|G| \leq \lfloor 42\sqrt{K_X^3 \cdot 7} \rfloor = \lfloor 168\sqrt{-21\chi(\mathcal{O}_X)} \rfloor.$$

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