On the Classification of Threefolds Isogenous to a Product

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Definition of the Objects

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$$X = (C_1 \times ... \times C_n)/G,$$

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$\Rightarrow$ X is smooth, minimal, of general type i.e. $\kappa(X) = n$ and K_X is ample

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Why shall we consider varieties isogenous to a product?

- find new examples of varieties of general type,
- interesting relations with group theory and computer algebra.

Beauvilles Example

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- $g(C) = \frac{1}{2}(5-1)(5-2) = 6 \quad \Rightarrow \quad \chi(\mathcal{O}_S) = \frac{(g(C)-1)^2}{25} = 1.$

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By the classical inequalities from surface geography

$$2p_g \leq K_S^2 \quad \text{if} \quad q \geq 1 \quad \text{(Debarre)} \quad \text{and} \quad K_S^2 \leq 9\chi(\mathcal{O}_S) = 9 \quad \text{(BMY)}.$$

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(see [CCML98, Pir02, HP02]).

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(As a first step in dimension three.)

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Theorem (Catanese)

Let $D_1, \ldots, D_k$ be pairwise non-isomorphic curves with $g(D_i) \geq 2$, then

$$\mathrm{Aut}\left(D_1^{n_1} \times \ldots \times D_k^{n_k}\right) = \left(\mathrm{Aut}(D_1)^{n_1} \rtimes \mathfrak{S}_{n_1}\right) \times \ldots \times \left(\mathrm{Aut}(D_k)^{n_k} \rtimes \mathfrak{S}_{n_k}\right)$$

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$$\text{Then} \quad G/G^0 \leq \mathfrak{S}_n, \quad \text{where} \quad G^0 := G \cap \left[\mathrm{Aut}(C_1) \times \ldots \times \mathrm{Aut}(C_n)\right].$$

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Definition

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certain kind of *combinatorial data:* the group G, the genera $g(C_i)$ etc.

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$\Rightarrow$ to go on, we need to understand faithful group actions on curves in greater detail

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$$\pi_1\left(C' \setminus \mathcal{B}, q_0\right) = \langle \gamma_1, \dots, \gamma_r, \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} \mid \gamma_1 \cdots \gamma_r \cdot \prod_{i=1}^{g'} [\alpha_i, \beta_i] \rangle.$$

A faithful group action $\psi\colon G \to \mathrm{Aut}(C)$ is given and completely determined by:

- a compact Riemann surface C',
- a finite set $\mathcal{B} \subset C'$ (the branch points) and
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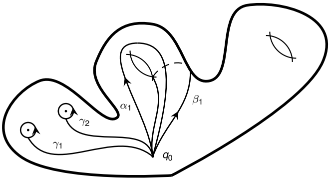

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$$\langle \chi, \chi_\varphi \rangle = \chi(1_G)(g' - 1) + \sum_{i=1}^{r} \sum_{\alpha=1}^{m_i - 1} \frac{\alpha \cdot N_{i,\alpha}}{m_i} + \langle \chi, \chi_{triv} \rangle.$$

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Output: a "finite list" of all threefolds X isogenous to a product with $\chi(\mathcal{O}_X) = \chi$.

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$\Rightarrow$ list of candidates for the numerical data

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- according to Hurwitz it holds $|G| \leq |\mathrm{Aut}(C_i)| \leq 84\big(g(C_i) - 1\big)$
- we conclude

$$-\chi(\mathcal{O}_X) = \frac{1}{|G|}\prod_{i=1}^{3}\big(g(C_i) - 1\big) \geq \frac{|G|^2}{84^3}.$$

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$\Rightarrow$ only finitely many numerical data (n, T_1, T_2, T_3)

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$$[G, T_1, T_2, T_3, h^{p,q}].$$

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Theorem (Frapporti,-)

Let X be a threefold isogenous to a product of unmixed type with $\chi(\mathcal{O}_X) = -1$. Then X is minimal of general type and there are 54 possibilities for

$$[G, T_1, T_2, T_3, h^{p,q}].$$

G	T_1	T_2	T_3	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{2,1}$
$\mathfrak{A}_5$	$[0; 2^3, 3]$	$[0; 2, 5^2]$	$[0; 3^2, 5]$	2	0	0	3	6
$GL(2, \mathbb{F}_3)$	$[0; 2, 3, 8]$	$[0; 2, 3, 8]$	$[2; -]$	5	5	2	11	17
$GL(2, \mathbb{F}_3)$	$[0; 2, 3, 8]$	$[0; 2, 3, 8]$	$[2; -]$	4	4	2	13	18
$\mathfrak{S}_4 \times \mathbb{Z}_2$	$[0; 2^5]$	$[0; 2, 4, 6]$	$[0; 2, 4, 6]$	3	1	0	5	9
$SL(2, \mathbb{F}_3)$	$[0; 3^2, 4]$	$[0; 3^2, 4]$	$[2; -]$	5	5	2	13	19
$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$[0; 2, 4, 6]$	$[0; 2, 4, 6]$	$[2; -]$	5	5	2	11	17
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$SD16$	$[0; 2, 4, 8]$	$[0; 2, 4, 8]$	$[2; -]$	5	5	2	11	17
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$\mathcal{D}_4 \times \mathbb{Z}_2$	$[0; 2^5]$	$[0; 2^3, 4]$	$[0; 2^3, 4]$	3	1	0	5	9
$\mathcal{D}_4 \times \mathbb{Z}_2$	$[0; 2^5]$	$[0; 2^3, 4]$	$[0; 2^3, 4]$	4	2	0	7	12
$Dic12$	$[0; 3, 4^2]$	$[0; 3, 4^2]$	$[2; -]$	5	5	2	13	19
$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$[0; 2, 6^2]$	$[0; 2, 6^2]$	$[2; -]$	6	6	2	11	18
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$\mathcal{D}_6$	$[0; 2^3, 3]$	$[0; 2^3, 6]$	$[1; 2^2]$	4	3	1	9	14
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$\mathcal{D}_6$	$[0; 2^5]$	$[0; 2^3, 3]$	$[1; 3]$	4	3	1	9	14
$\mathbb{Z}_{10}$	$[0; 2, 5, 10]$	$[0; 2, 5, 10]$	$[2; -]$	5	5	2	13	19
$\mathbb{Z}_{10}$	$[0; 2, 5, 10]$	$[0; 2, 5, 10]$	$[2; -]$	6	6	2	11	18
$\mathbb{Z}_{10}$	$[0; 2, 5, 10]$	$[0; 2, 5, 10]$	$[2; -]$	4	4	2	15	20

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Q	$[0;4^3]$	$[0;4^3]$	$[2;-]$	5	5	2	13	19
$\mathbb{Z}_8$	$[0;2,8^2]$	$[0;2,8^2]$	$[2;-]$	6	6	2	11	18
$\mathbb{Z}_8$	$[0;2,8^2]$	$[0;2,8^2]$	$[2;-]$	4	4	2	15	20
$\mathcal{D}_4$	$[0;2^3,4]$	$[1;2]$	$[1;2^2]$	4	4	2	11	16
$\mathcal{D}_4$	$[0;2^3,4]$	$[0;2^2,4^2]$	$[1;2^2]$	4	3	1	9	14
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$\mathcal{D}_4$	$[0;2^6]$	$[0;2^3,4]$	$[1;2]$	4	3	1	9	14
$\mathbb{Z}_2^3$	$[0;2^5]$	$[0;2^5]$	$[0;2^5]$	5	3	0	9	15
$\mathbb{Z}_2^3$	$[0;2^5]$	$[0;2^5]$	$[0;2^5]$	4	2	0	7	12
$\mathbb{Z}_6$	$[0;3,6^2]$	$[0;3,6^2]$	$[2;-]$	6	6	2	11	18
$\mathbb{Z}_6$	$[0;3,6^2]$	$[0;3,6^2]$	$[2;-]$	4	4	2	15	20
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- there are 108 examples in the mixed case

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Aim: study the geography of $\widehat{X}$ i.e. the relations between the Chern invariants

$$\chi(\mathcal{O}_{\widehat{X}}), \qquad e(\widehat{X}) \qquad \text{and} \qquad K_{\widehat{X}}^3.$$

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locally X is a quotient of $\mathbb{C}^3$ by a diagonal linear automorphism

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- there are finitely many points (x, y, z) on $C_1 \times C_2 \times C_3$ with non-trivial stabilizer.
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$\Rightarrow$ we can explicitly compute $\widehat{X}$ and derive relations between the Chern invariants $\chi(\mathcal{O}_{\widehat{X}})$, $e(\widehat{X})$ and $K^3_{\widehat{X}}$.

Product quotient threefolds

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- in the case where $\widehat{X}$ is smooth, we have a way to determine the Hodge numbers of $\widehat{X}$ and an algorithm to classify these varieties for a fixed value of $\chi(\mathcal{O}_{\widehat{X}})$ in the sense above.

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