Stokes Solvers for Variable Viscosity Mantle Convection

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Viscosity in the Earth's mantle varies by several orders of magnitude, dependent on temperature, pressure, grain size and phase transformations. Modeling these variations realistically in three-dimensional mantle convection simulations has been a long-standing problem. We present various approaches to increase the robustness and efficiency of the solution process in a 2-D rectangular domain as well as in a spherical shell.

A stabilization of the finite-element discretization with piecewise linear resp. bilinear trial and test functions for both, velocity and pressure, is presented and evaluated in terms of matrix properties of the Schur complement. The stabilization method uses polynomial pressure projections and is described and analyzed in [1]. The resulting discrete Stokes system fulfills a generalized inf-sup condition with a grid-independent inf-sup constant.

Viscosity dependence of the matrix entries in the momentum operator and in the Schur complement is removed by diagonal scaling as is done in [2]. Here, a viscosity scaled pressure mass matrix as in [3] is used. A preconditioner is then necessary only for the momentum operator $\mathbf{A}$ having a condition number proportional to the number of grid points. This indicates the use of a multigrid method which utilizes the low condition numbers on coarse grids efficiently. Such a method is used in the 3-D spherical code TERRA. For the scaled Schur complement no further preconditioning is necessary so that a CG method can be used to solve for pressure. With applying velocity corrections from the $\mathbf{A}^{-1}$-evaluation of the application of the Schur complement, this CG method is named after Uzawa, who gave the idea in [4]. Another idea, taken from [5] is to restart the whole Uzawa algorithm when the $\mathbf{A}^{-1}$-evaluation cannot be done with sufficient accuracy. Therewith, and with suitable stopping criteria, which are derived from eigenvalue estimates, the Uzawa method shows impressive robustness w.r.t. viscosity variations and in most cases outperforms a preconditioned MINRES method which has been implemented for comparison. However, as the pressure error is now reduced in a viscosity-dependent norm, induced by the above-mentioned mass matrix, the residual has to be reduced below a lower threshold than in the case of using a standard mass matrix. This drawback is far outweighted by the much better convergence when the viscosity-dependent mass matrix is used.

References

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