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CODIMENSION TWO BIFURCATION IN A CONVECTION EXPERIMENT[1]

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ABSTRACT. In a binary fluid mixture convection sets in either as a steady or oscillatory flow, depending on the sign and magnitude of the separation ratio. There exists a critical separation ratio where the two instability lines meet. This is an example of a codimension two bifurcation. In this paper measurements are presented which show qualitative agreement with a number of predictions for the vicinity of that bifurcation.

The experiments are done with a rectangular Rayleigh-Bénard cell of aspect ratio 1-4-8, filled with a normal-fluid mixture of ^{3}He and ^{4}He embedded in a porous medium. The top plate of the cell is regulated at a constant temperature, while the bottom is heated with constant power. The temperature of the bottom plate is then measured as a function of that power. The apparatus and experimental procedure are described in Ref. 1, where references to the theoretical background are given as well.

Figure 1 summarizes the theoretical predictions for the onset of convection as a function of the separation ratio ψ in the neighborhood of the codimension-two point. At the solid line the motionless state becomes unstable with respect to steady convection rolls. This bifurcation is forward for $\psi > \psi_t$ where ψ_t has a small negative value. For $\psi < \psi_t$ this bifurcation is backward. The hysteresis connected with the backward bifurcation terminates at the dotted saddle-node line, where the jump to the ground state occurs when the Rayleigh number is decreased. For ψ somewhat smaller than ψ_t, the first instability to occur is an oscillatory one. The dashed line showing the location of this Hopf bifurcation terminates at the steady bifurcation line. The point where these two lines meet is the codimension two point. The frequency of the oscillation goes to zero when approaching that point. The two stable convecting steady solutions (which exist above the dotted saddle node line) are separated by two unstable steady solutions when the Rayleigh number is below its value for the steady bifurcation (solid line). This causes a fourth (dashed-dotted) bifurcation line: when the amplitude of the oscillation about the motionless state is large

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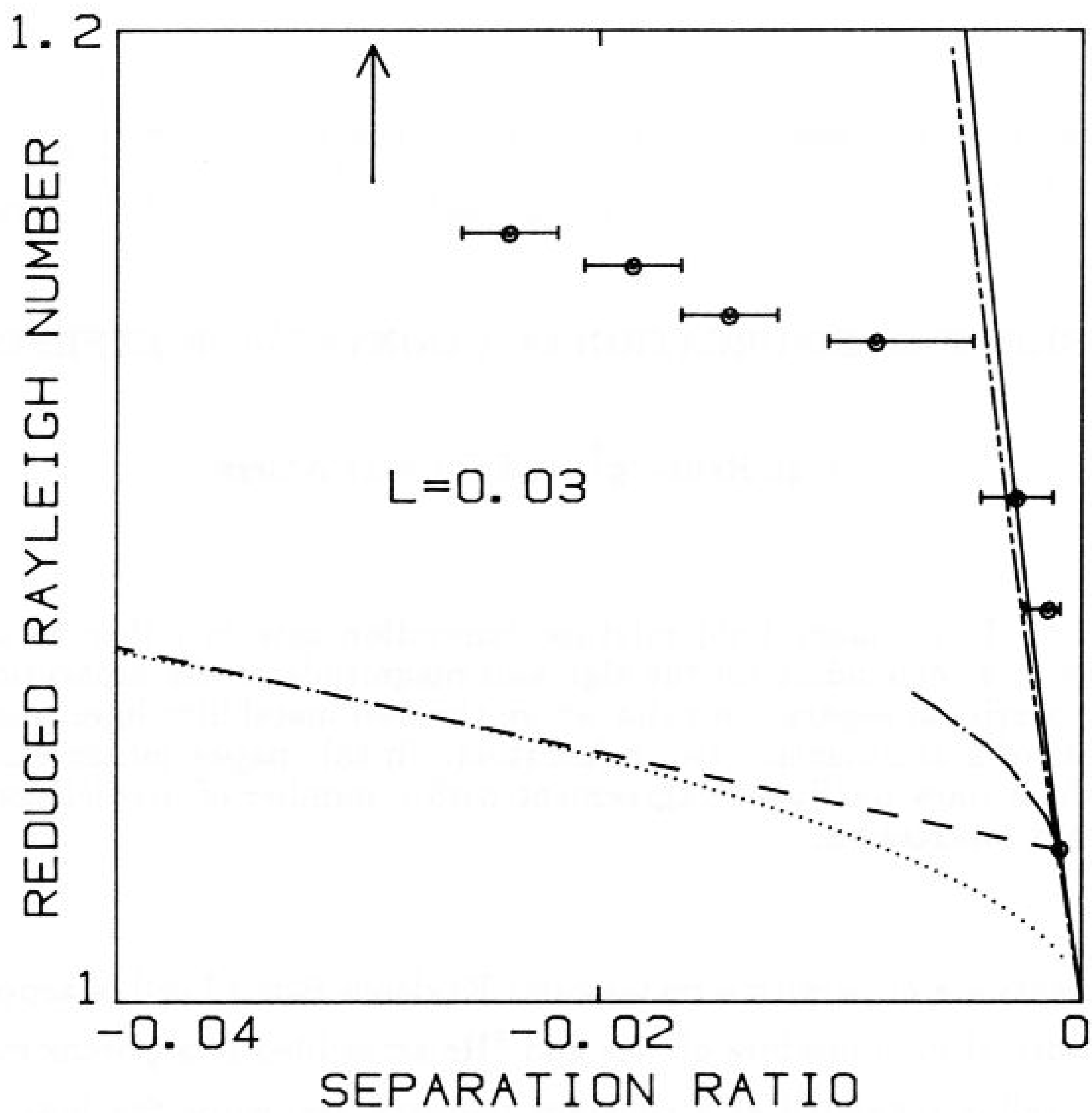

Figure 1. Stability diagram of the mixture. The solid line gives the bifurcation to steady convection, and the dashed line gives the Hopf bifurcation. The dotted curve corresponds to the saddle-node line calculated from a Lorenz-like five-mode truncated model. The dash-dotted line gives the location of the heteroclinic orbit of that model. The data points are the experimental results for the heteroclinic orbit.

enough to reach the unstable solutions, a heteroclinic orbit connecting the two unstable fixed points is formed. Approaching that orbit is accompanied by a vanishing frequency. Crossing it results in a hysteretic transition to one or the other of the steady convecting states. The two finite amplitude instabilities (saddle-node and heteroclinic orbit) are not predictable by means of a linear stability analysis of the motionless state. We used a truncated model similar to that given by Veronis[2] to calculate those lines. Details of that calculation will be presented elsewhere. The heteroclinic orbit line terminates because the oscillatory solution of the model becomes unstable against a period-doubling cascade into chaos at that point, as reported by Da Costa *et al.* [3]

In our experiment ψ can be varied from $-\infty$ to some positive value by changing the mean temperature of the fluid. An increase of the bottom temperature with regulated top temperature then corresponds to an increase of the Rayleigh number with (almost) constant ψ. Figure 2 shows time series of such an experiment for a top temperature of 2.2064 K ($\psi \cong -0.01$). Below and up to a temperature difference of 6.364 mK no oscillations of the bottom temperature are observed. They set in via a non-hysteretic Hopf bifurcation when ΔT is increased to 6.371 mK. Further increase of ΔT increases the period

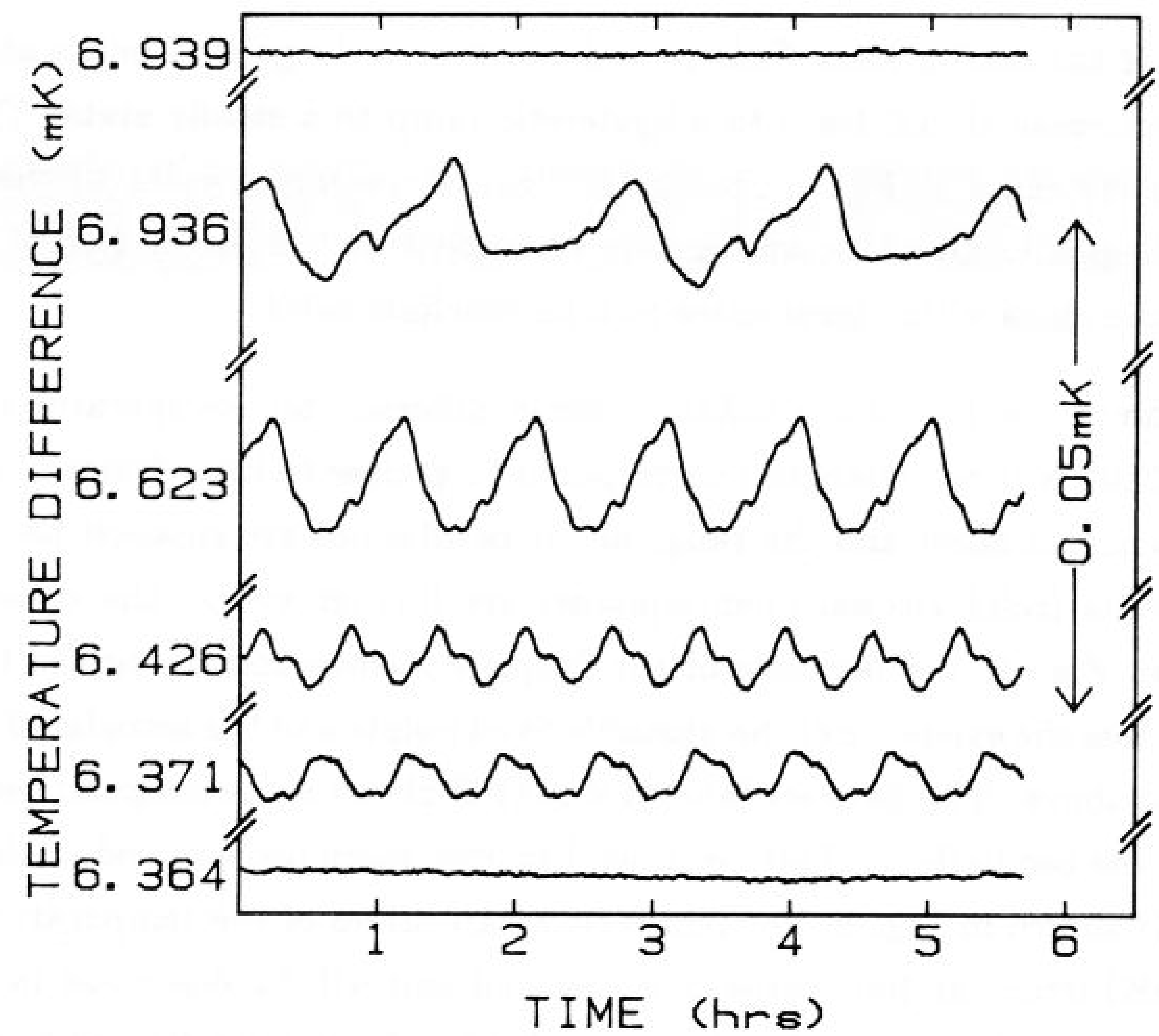

Figure 2. The temperature difference across the cell, with the bottom plate heated at constant power, for six different values of that power. The numbers on the left indicate the mean temperature differences between top and bottom plate. The top temperature is 2.2064 K. The run corresponds to the third point to the left of the codimension two point in Fig. 1.

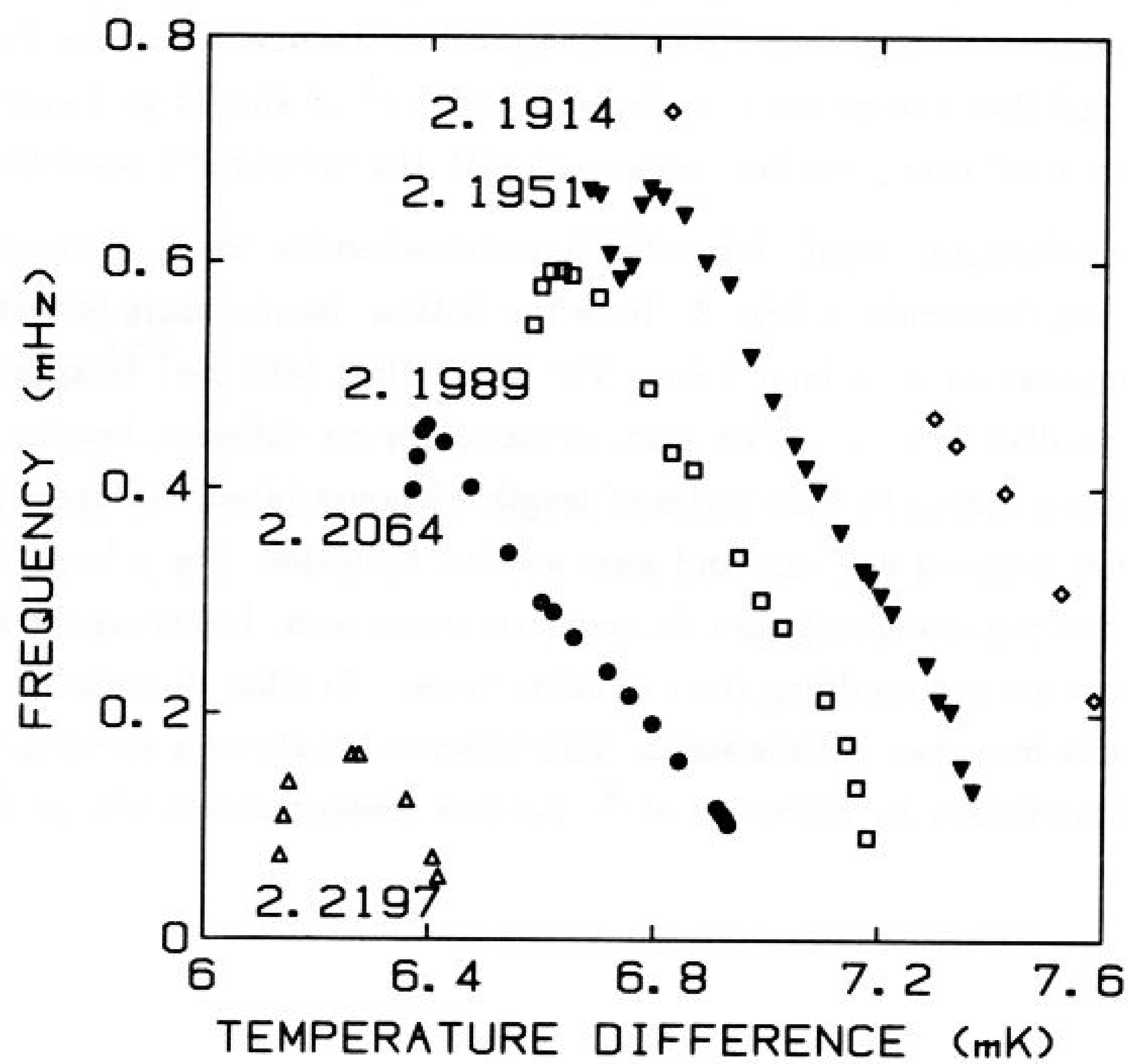

Figure 3. The frequency of the temperature oscillations of the bottom plate as a function of the mean temperature difference for five different top temperatures. The numbers are the temperature of the top in Kelvin.

and amplitude of the oscillations. This period becomes as large as 3 hours at 6.936 mK. A tiny further increase of ΔT leads to a hysteretic jump to a steady state. The location of that jump is indicated in Fig. 1, being the 3rd measurement point to the left of the codimension-two point. Our observations are consistent with a forward Hopf bifurcation to a time periodic state which terminates in a heteroclinic orbit.

Figure 3 shows the frequency behavior for 5 different top temperatures. The top temperature 2.2197 K (open triangles) corresponds to ψ close to its codimension-two value. Both the frequency at onset and the range of the oscillations are smallest here. The next two measurements (solid circles, open squares) are further away: the onset frequency increases. As in Fig. 2, the decrease of the frequency when increasing the temperature difference indicates the existence of the unstable fixed points and the associated heteroclinic orbits discussed above. The data set labeled 2.1951 K shows some irregular behavior close to the onset of the oscillations. This behavior becomes more pronounced in the last set of data (2.1914 K) shown in Fig. 3. For relatively small values of the temperature difference ($\Delta T \leq 7.23$ mK) irregular time series are obtained and will be described in more detail in Fig. 5. Decreasing the top temperature even further (more negative ψ) to 2.1877 K no longer leads to the oscillations described in Fig. 2. A hysteretic instability to a convecting state occurs. Time dependence of that convecting state is characterized by frequencies which are higher than those shown in Figs. 2 to 4.

Figure 4 shows the frequencies ω_c measured at the onset of the oscillations as a function of the top temperature. These frequencies correspond to the ones along the dashed line in Fig. 1. Linear stability theory predicts that the value of ω_c should go to zero when the codimension-two point is approached, consistent with the measured frequencies.

Figure 5 provides some insight into what happens when the top temperature is 2.1914 K, as for the open diamonds in Fig. 3. Here the bottom temperature is plotted against the bottom temperature at a later time. The delay time (650 sec) is approximately a quarter of the oscillation period. Six measurements for six different heating powers are shown, each corresponding to time series of length 11 hours (about 15 cycles). This plot allows distinction between periodic and non-periodic behavior. For a large temperature difference (the two upper curves) periodic oscillations are seen. Lowering the temperature difference leads to period doubling (two middle curves). Further decrease of the heating power then results in a chaotic time series. This observation shows a striking similarity to the theoretical prediction by Moore *et al.* [4] Further measurements will be done on this topic.

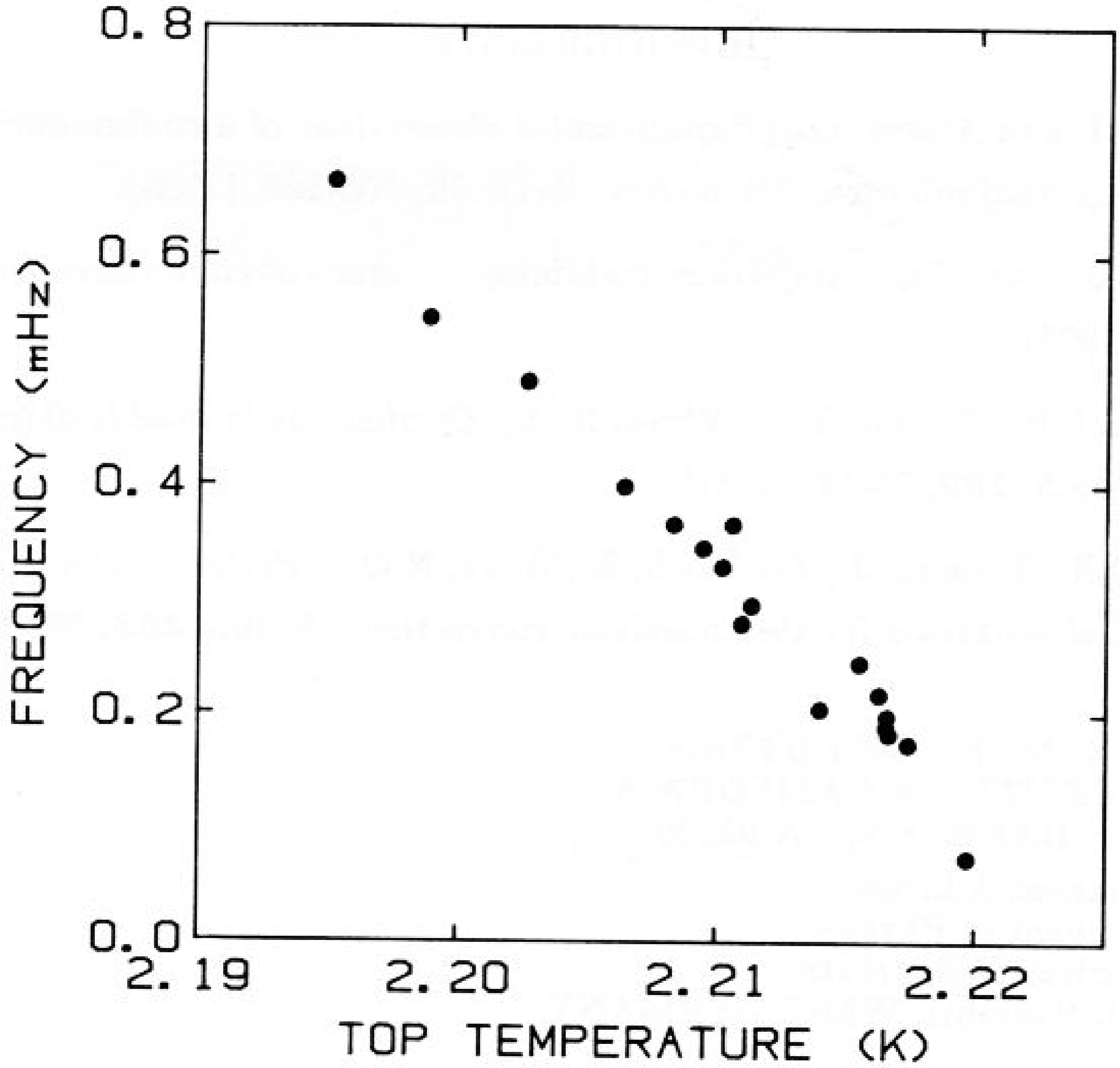

Figure 4. The frequency measured at the onset of the oscillations as a function of the top temperature.

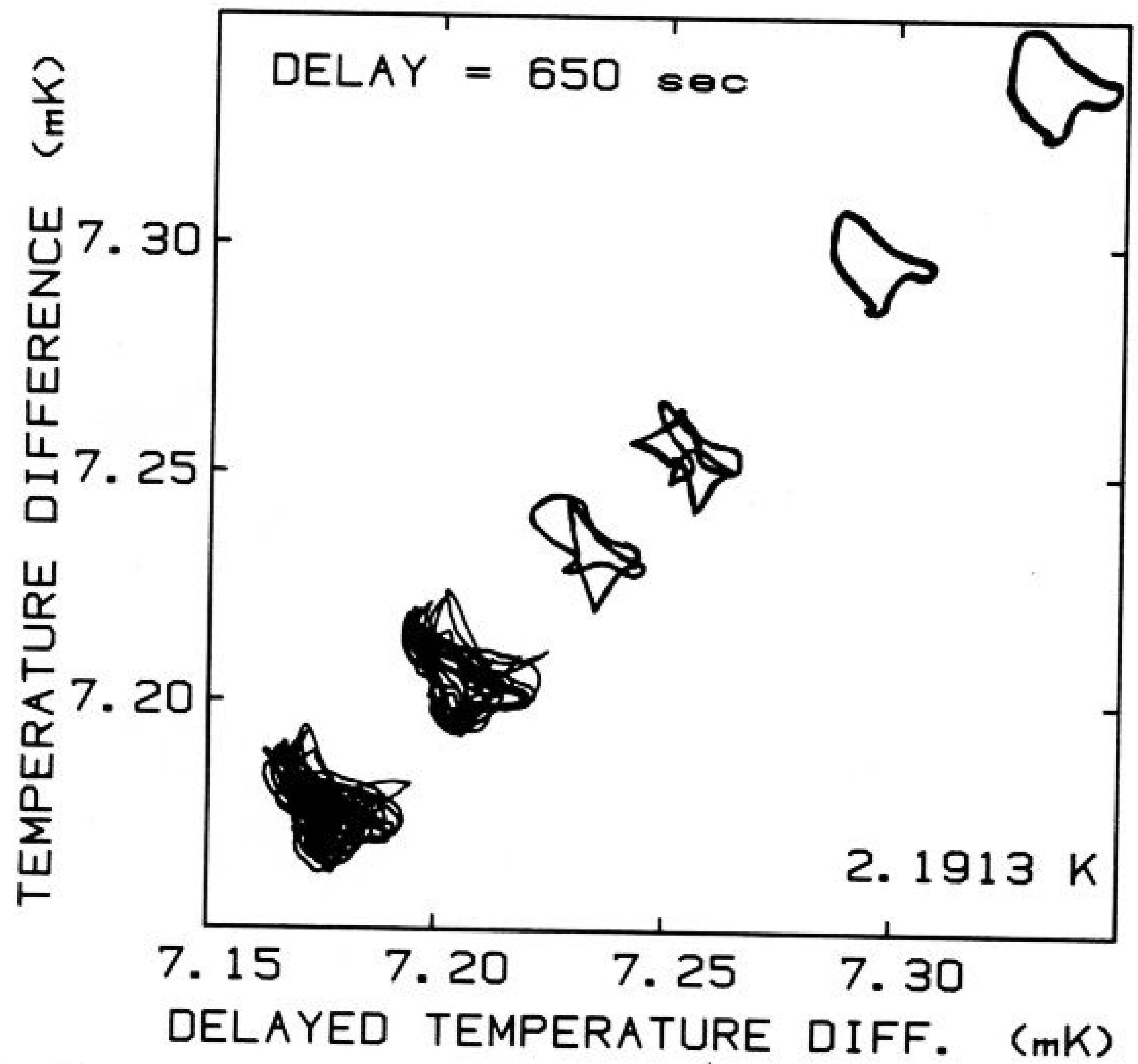

Figure 5. The temperature difference across the cell, plotted against the temperature difference at a later time. Six time series are represented. The two near the upper right (largest ΔT) show periodic behavior. The middle two correspond to a period-doubled state. The two in the lower left (smallest ΔT) represent chaotic behavior.

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