Spectral-finite element approach to present-time mantle convection

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Summary 1

We present a Spectral-Finite Element (SFE) approach to the forward modeling of present-time mantle convection. The differential Stokes problem for an incompressible viscous flow in a spherical shell is reformulated in weak sense by means of a variational principle. The integral equations obtained are then parametrized by vector and tensor spherical harmonics in the angular direction and by piecewise linear finite elements over the radial direction. The solution is obtained using the Galerkin method, that leads to the solution of a system of linear algebraic equations. The earth-viscosity structure is described using a two-dimensional spherical grid, that allows us to treat various kinds of lateral variation, with viscosity contrasts of several order of magnitude.

The method is first tested for the case of a one-dimensional viscosity structure. After prescribing the internal load in the form of a Dirac-delta, Green's functions for surface topography, core topography and geoid are computed and compared with those obtained by solving the problem with the traditional matrix-propagator technique. As example, we report here Green's functions for geoid obtained for a uniformly viscous and for a two-layer mantle.

The approach is then applied to two different axisymmetric viscosity structures consisting either of one or two highly viscous cratonic bodies embedded in the upper mantle. We compute the corresponding Green's functions, showing the non-linear coupling of various spherical harmonic modes, and the resulting angular dependence of vertical flow and stress.

Differential formulation 2

Continuity equation for an *incompressible* mantle:

$$\mathrm{div}\boldsymbol{v} = 0 \text{ in } B,$$

where $\boldsymbol{v} = \boldsymbol{v}(r,\Omega)$ is the flow velocity and $B = B_{LM} \cup B_{UM}$ (see Fig. 4) the mantle volume.

Momentum equation under the approximation of infinite Prandtl number:

$$\mathrm{div}\boldsymbol{\tau} + \boldsymbol{f} = \boldsymbol{0} \text{ in } B,$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}(r,\Omega)$ is the Cauchy stress tensor and $\boldsymbol{f} = \Delta\varrho\,\boldsymbol{g}$ is the body force given by the product of the density perturbations $\Delta\varrho = \Delta\varrho(r,\Omega)$ and the reference gravitational acceleration $\boldsymbol{g} = \boldsymbol{g}(r)$.

Constitutive relation for a *Newtonian continuum*:

$$\boldsymbol{\tau} = -p\boldsymbol{I} + 2\eta\boldsymbol{\varepsilon}$$

where $\boldsymbol{I}$ is the identity tensor, $p = p(r,\Omega)$ is the pressure, $\eta = \eta(r,\Omega)$ is the dynamic viscosity and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(r,\Omega)$ is the strain-rate tensor.

Boundary conditions for *impermeable* and *free-slip* mantle boundaries:

$$\left.\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{n} &= 0 \\ \boldsymbol{\tau}\cdot\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{\tau}\cdot\boldsymbol{n})\boldsymbol{n} &= \boldsymbol{0} \end{aligned}\right\} \quad \text{on } \partial B$$

where $\partial B = \partial B^a \cup \partial B^c$ (see Fig. 4), the superscripts a and c denoting respectively the Earth's surface and the core-mantle boundary (cmb), and $\boldsymbol{n}$ is the outward normal to those surfaces.

Weak formulation 3

On the space $\mathcal{V} = \{\boldsymbol{v} \in \mathcal{W}^1(B);\ \lambda^1 \in \mathcal{L}^2(B);\ \lambda^i \in \mathcal{L}^2(\partial B),\ i = 2,3\}$, where $\mathcal{L}^2(B)$ and $\mathcal{L}^2(\partial B)$ are the space of square-integrable scalar functions in B and on ∂B respectively, $\mathcal{W}^1(B)$ is the Sobolev space of vector functions in B, i.e. $\mathcal{W}^1(B) = \{\boldsymbol{v}, \mathrm{grad}\,\boldsymbol{v} \in \mathcal{L}^2(B)\}$, the following functional is introduced:

$$F(\boldsymbol{v},\boldsymbol{\lambda}) = \int_B \eta(\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon})\,dV + \int_B \boldsymbol{f}\cdot\boldsymbol{v}\,dV + \int_B \lambda^1\,\mathrm{div}\,\boldsymbol{v}\,dV + \int_{\partial B^a} \lambda^2(\boldsymbol{n}\cdot\boldsymbol{v})\,dS + \int_{\partial B^c} \lambda^3(\boldsymbol{n}\cdot\boldsymbol{v})\,dS\,,$$

where $\boldsymbol{\lambda} = (\lambda^1,\lambda^2,\lambda^3)$ is a vector of Lagrange multipliers that represent the pressure, the radial stress at the Earth's surface and at the cmb, respectively. By computing the variation of F and making use of the Green's identity, we obtain

$$\delta F(\boldsymbol{v},\boldsymbol{\lambda},\delta\boldsymbol{v},\delta\boldsymbol{\lambda}) = -\int_B (\mathrm{div}\,\boldsymbol{\tau} + \boldsymbol{f})\cdot\delta\boldsymbol{v}\,dV + \int_B (\mathrm{div}\,\boldsymbol{v})\,\delta\lambda^1\,dV + \int_{\partial B^a} (\boldsymbol{n}\cdot\boldsymbol{v})\,\delta\lambda^2\,dS + \int_{\partial B^c} (\boldsymbol{n}\cdot\boldsymbol{v})\,\delta\lambda^3\,dS + \int_{\partial B} [\boldsymbol{\tau}\cdot\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{\tau}\cdot\boldsymbol{n})\boldsymbol{n}]\cdot\delta\boldsymbol{v}\,dS\,.$$

The weak formulation of the problem consists of finding $(\boldsymbol{v},\boldsymbol{\lambda}) \in \mathcal{V}$ such that $\delta F = 0\ \forall\,(\delta\boldsymbol{v},\delta\boldsymbol{\lambda}) \in \mathcal{V}$.

Parameterization 4

Over the angular coordinates $(\vartheta,\varphi) = \Omega$, the fields λ^i, their variations $\delta\lambda^i$ and $\Delta\varrho$ are expanded into scalar spherical harmonics $Y_{jm}(\Omega)$, e.g.:

$$\lambda^1(r,\Omega) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} \Lambda^1_{jm}(r)Y_{jm}(\Omega)\,,$$

while $\boldsymbol{v}$, its variation $\delta\boldsymbol{v}$ and $\boldsymbol{f}$ are expanded into vector spherical harmonics $\boldsymbol{S}^{(\ell)}_{jm}$, $\ell = -1,0,1$, e.g.:

$$\boldsymbol{v}(r,\Omega) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\left[U_{jm}(r)\boldsymbol{S}^{(-1)}_{jm}(\Omega) + V_{jm}(r)\boldsymbol{S}^{(1)}_{jm}(\Omega) + W_{jm}(r)\boldsymbol{S}^{(0)}_{jm}(\Omega)\right].$$

Over the radial range $[c,a]$ (Fig. 1), the radially dependent part of each field is expanded into a finite combination of piecewise linear functions $\psi_k(r_i) = \delta_{ki}$ that form a base of the space $\mathcal{W}^1(c,a)$, e.g.:

$$U_{jm}(r) = \sum_{k=1}^{P+1} U^k_{jm}\psi_k(r)\,.$$

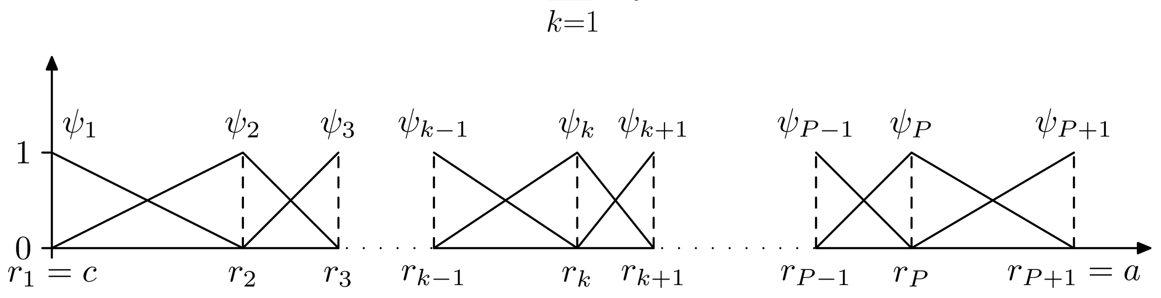

Figure 1. Piecewise linear functions over $[c,a]$. On the interval $r_k \le r \le r_{k+1}$ only two functions are non-zero, namely $\psi_k(r) = \frac{r_{k+1}-r}{r_{k+1}-r_k}$ and $\psi_{k+1}(r) = \frac{r-r_k}{r_{k+1}-r_k}$.

Existence and uniqueness 5

Necessary conditions for the *existence* of the solution of the problem are the *conditions of equilibrium* (the resultant of forces as well as of moments vanish):

$$\int_B \boldsymbol{f}\,dV + \int_{\partial B} \boldsymbol{T}\,dS = \boldsymbol{0}$$

$$\int_B (\boldsymbol{r}\times\boldsymbol{f})\,dV + \int_{\partial B} (\boldsymbol{r}\times\boldsymbol{T})\,dS = \boldsymbol{0}\,,$$

where $\boldsymbol{T} = \boldsymbol{n}\cdot\boldsymbol{\tau}$ is the stress vector. After expanding $\boldsymbol{\tau}$ in tensor spherical harmonics $\boldsymbol{Z}^{(\ell)}_{jm}$, $\ell = 1,\ldots,6$,

$$\boldsymbol{\tau}(r,\Omega) = \sum_{\ell=1}^{6}\sum_{j=0}^{\infty}\sum_{m=-j}^{j} \tau^{(\ell)}_{jm}\boldsymbol{Z}^{(\ell)}_{jm}\,,$$

the existence conditions are reduced to the following spectral form:

$$\sum_{m=-1}^{1}\left[\int_{r=c}^{r=a} g\Delta\varrho_{1m}r^2 dr\right]\boldsymbol{e}_m = \sum_{m=-1}^{1}\left[a^2(\tau_{rr})_{1m}(a) - c^2(\tau_{rr})_{1m}(c)\right]\boldsymbol{e}_m$$

$$\sum_{m=-1}^{1}\left[c^3(\tau_{r\varphi})_{1m}(c) - a^3(\tau_{r\varphi})_{1m}(a)\right]\boldsymbol{e}_m = \boldsymbol{0}\,,$$

where $\boldsymbol{e}_m$, $m = -1,0,1$, are cyclic covariant base vectors. The first equation has been checked numerically, while the second is automatically verified because of the absence of surface and internal toroidal forcing.

The *uniqueness* of the solution is guaranteed by the mixed character of the boundary conditions (on both domain's boundaries, a Dirichlet condition on $\boldsymbol{v}$ and a Neumann condition on $\boldsymbol{\tau}$ are prescribed). Furthermore, the non-uniqueness of the solution has been proved in the case of purely Neumann boundary conditions.

Green's functions for 1D viscosity model 6

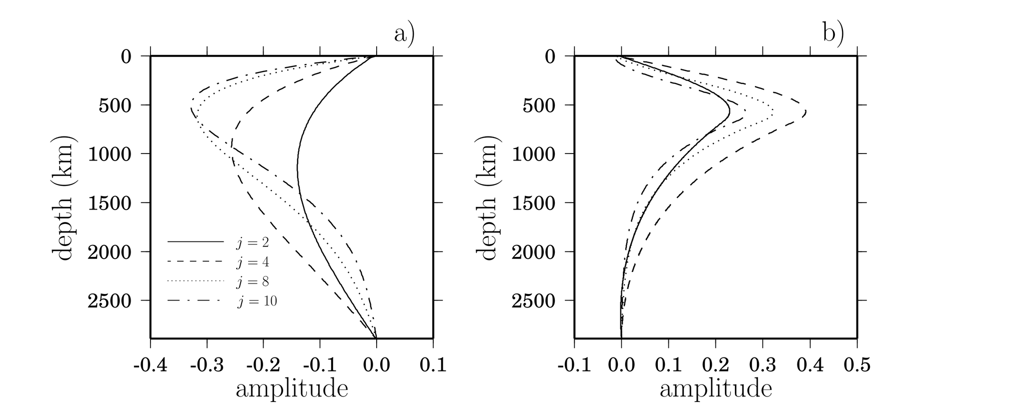

Figure 2. Green's functions for geoid obtained with SFE technique for different harmonic degrees and the following viscosity models: a) $\eta_{LM}/\eta_{UM} = 1$, b) $\eta_{LM}/\eta_{UM} = 100$.

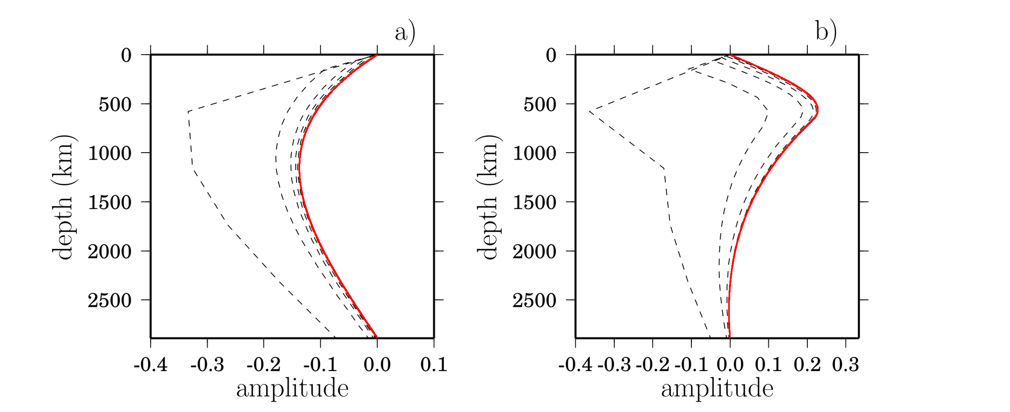

Figure 3. Convergence of the SFE method. Green's functions for geoid of degree $j = 2$ and viscosity model of Fig. 2. Dashed lines indicate the SFE solution for increasing number of finite elements (P =5, 20, 50, 200, 400), while the red line shows the analytical solution obtained with matrix-propagator technique.

Geometry of 2D viscosity model 7

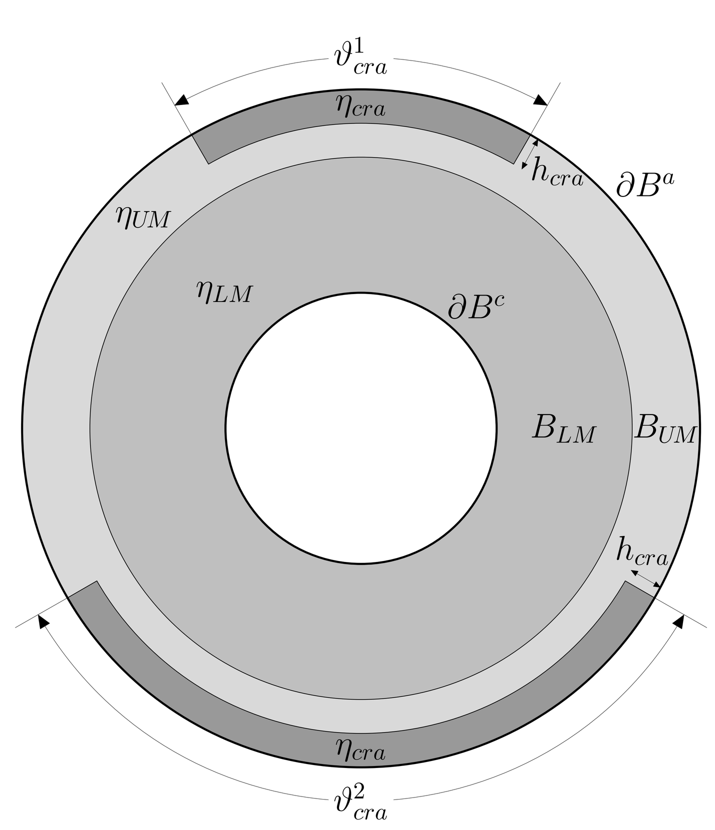

Figure 4. Two-dimensional axisymmetric geometry of the model. The whole mantle volume B, whose boundaries are denoted by ∂B^c and ∂B^a, is divided into a lower mantle B_{LM} of viscosity η_{LM} and an upper mantle B_{UM} of viscosity η_{UM}. Two cratonic structures of high viscosity η_{cra}, depth h_{cra} and angular amplitudes ϑ^1_{cra} and ϑ^2_{cra} are imbedded in the upper mantle.

Green's functions for 2D viscosity model 8

In the case of two-dimensional viscosity structure, spherical harmonic modes are no longer decoupled. An internal loading of given harmonic degree produces response on the whole spectrum.

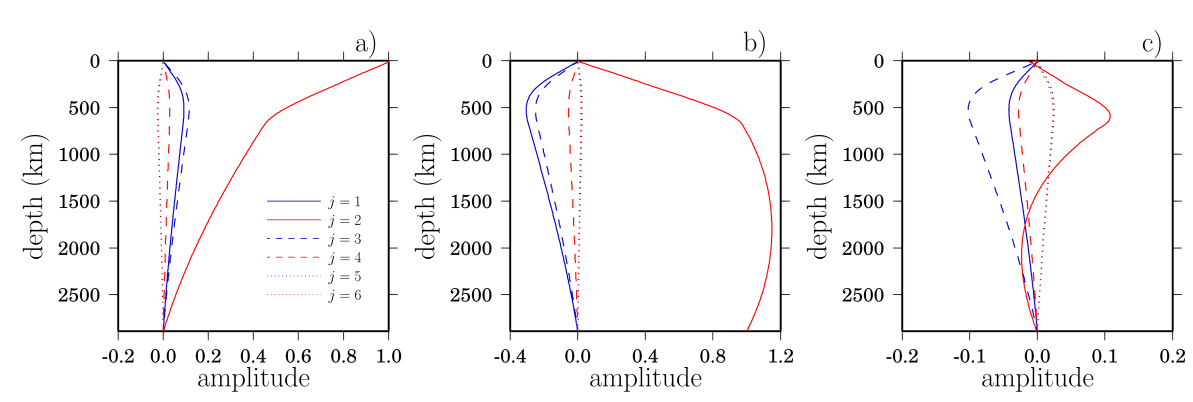

Figure 5. Green's functions for a) surface topography, b) core topography and c) geoid for a second-degree load. The results apply to the model with only the north-pole craton ($\vartheta^2_{cra} = 0$) and $\eta_{LM} = 10^{22}$ Pa s, $\eta_{UM} = 10^{20}$ Pa s, $\eta_{cra} = 10^{25}$ Pa s, $h_{cra} = 200$ km, $\vartheta^1_{cra} = 40°$.

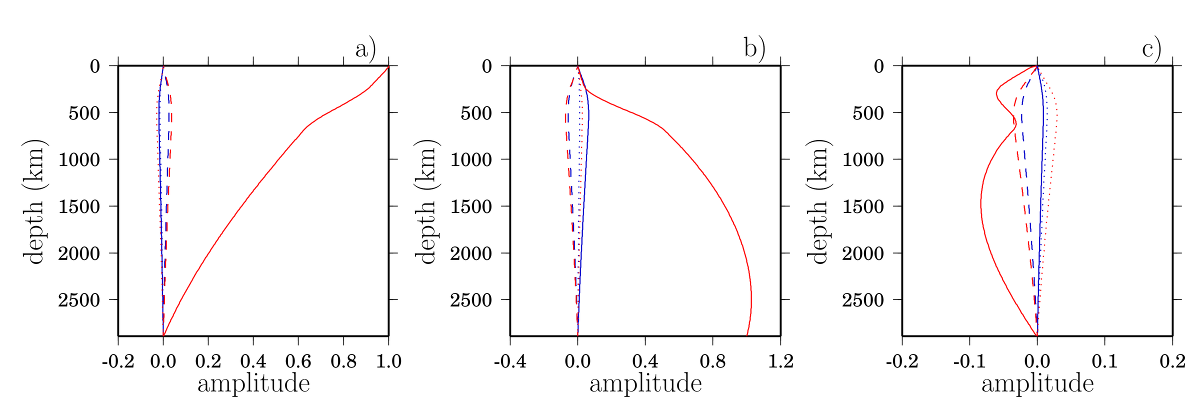

Figure 6. As in Fig. 5, except for the presence of a south-pole craton of angular amplitude $\vartheta^2_{cra} = 50°$.

Angular dependence of vertical flow and stress 9

A delta-like second-degree load is placed in the upper mantle at a depth of 480 km. We investigate the response for vertical flow and vertical stress at the same depth. While the flow tends to be blocked by the highly viscous cratons, the stress increases there, simulating a possible behaviour of the elastic lithosphere.

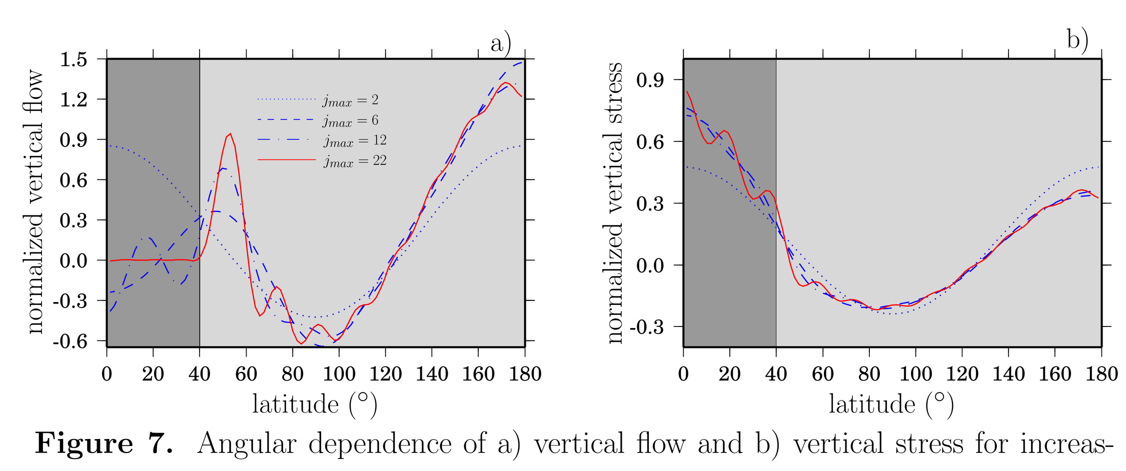

Figure 7. Angular dependence of a) vertical flow and b) vertical stress for increasing cut-off degree j_{max} for the following model: $\eta_{LM} = 10^{22}$ Pa s, $\eta_{UM} = 10^{20}$ Pa s, $\eta_{cra} = 10^{25}$ Pa s, $h_{cra} = 200$ km, $\vartheta^1_{cra} = 40°$, $\vartheta^2_{cra} = 0°$. Dark gray indicates the position of the craton, while light gray the upper mantle.

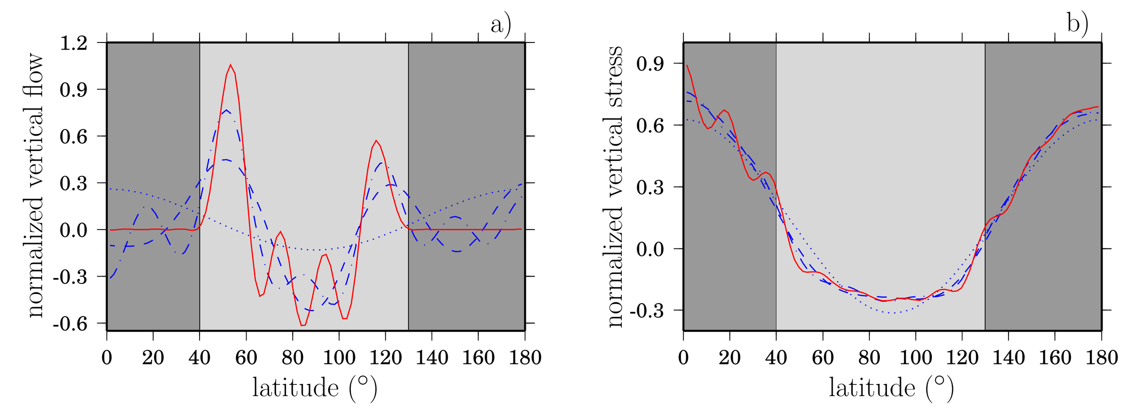

Figure 8. As in Fig. 7, except for the presence of a south-pole craton of angular amplitude $\vartheta^2_{cra} = 50°$.