Multi-Scale Methods for Elliptic and Parabolic Problems with Strongly Varying Coefficients

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Outline

- Targeted problems
- Upscaling vs. multi-scale modeling
- Brief review of MS methods for elliptic problems
- MSFV method for elliptic and parabolic problems
- Example and discussion

Targeted Problems

$$\alpha p \; + \; \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} \; = \; L(p) \; = \; q$$

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$$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$

reservoir simulation:

$$C\frac{p^{n+1} - p^n}{\Delta t} - \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p^{n+1}}{\partial x_j}\right\} = Q$$

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reservoir simulation:

$$C\frac{p^{n+1}-p^n}{\Delta t} - \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p^{n+1}}{\partial x_j}\right\} = Q$$

equivalent to:

$$\frac{C}{\Delta t}p^{n+1} + \frac{\partial}{\partial x_i}\left\{-\lambda_{ij}\frac{\partial p^{n+1}}{\partial x_j}\right\} = Q + \frac{C}{\Delta t}p^n$$

Targeted Problems

$$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$

incompressible Navier Stokes (pressure Poisson equation):

$$\frac{\partial}{\partial x_i}\left\{\frac{\partial p}{\partial x_i}\right\} = -\rho\frac{\partial}{\partial x_i}\left\{\frac{\partial u_i u_j}{\partial x_j}\right\}$$

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$$\frac{\partial}{\partial x_i}\left\{\frac{\partial p}{\partial x_i}\right\} = -\rho\frac{\partial}{\partial x_i}\left\{\frac{\partial u_i u_j}{\partial x_j}\right\}$$

equivalent to:

$$\frac{\partial}{\partial x_i}\left\{\delta_{ij}\frac{\partial p}{\partial x_j}\right\} = -\rho\frac{\partial}{\partial x_i}\left\{\frac{\partial u_i u_j}{\partial x_j}\right\}$$

Targeted Problems

$$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$

Stokes:

$$\frac{\partial}{\partial x_i}\left\{\delta_{ij}\frac{\partial p}{\partial x_j}\right\} = 0$$

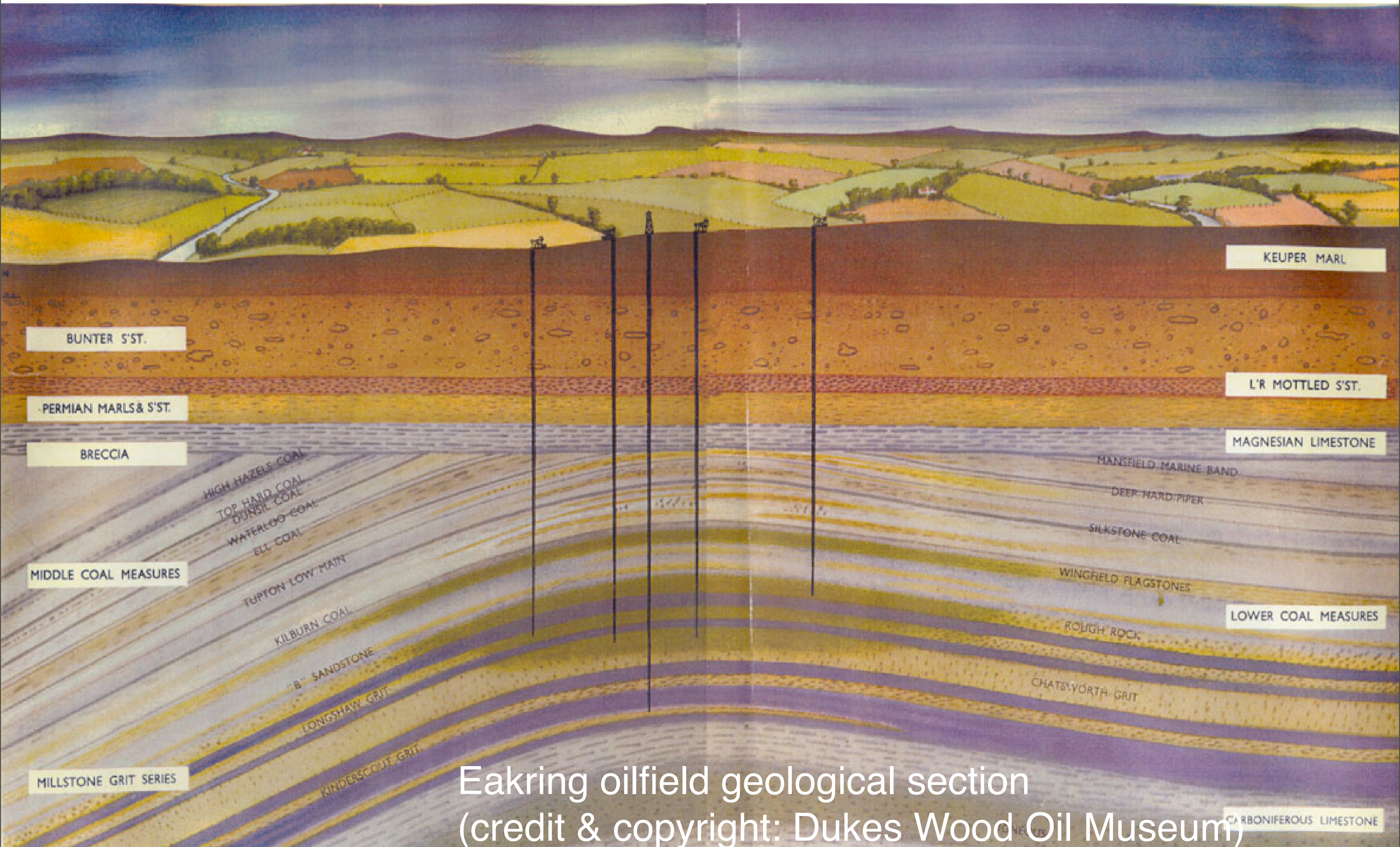
$$\frac{\partial}{\partial x_i}\left\{\mu\delta_{ij}\frac{\partial u_k}{\partial x_j}\right\} = \frac{\partial p}{\partial x_k}$$

Targeted Problems

$$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$

electrodynamics, ...

Major Challenge: Multi-Scale Coefficients



Eakring oilfield geological section
(credit & copyright: Dukes Wood Oil Museum)

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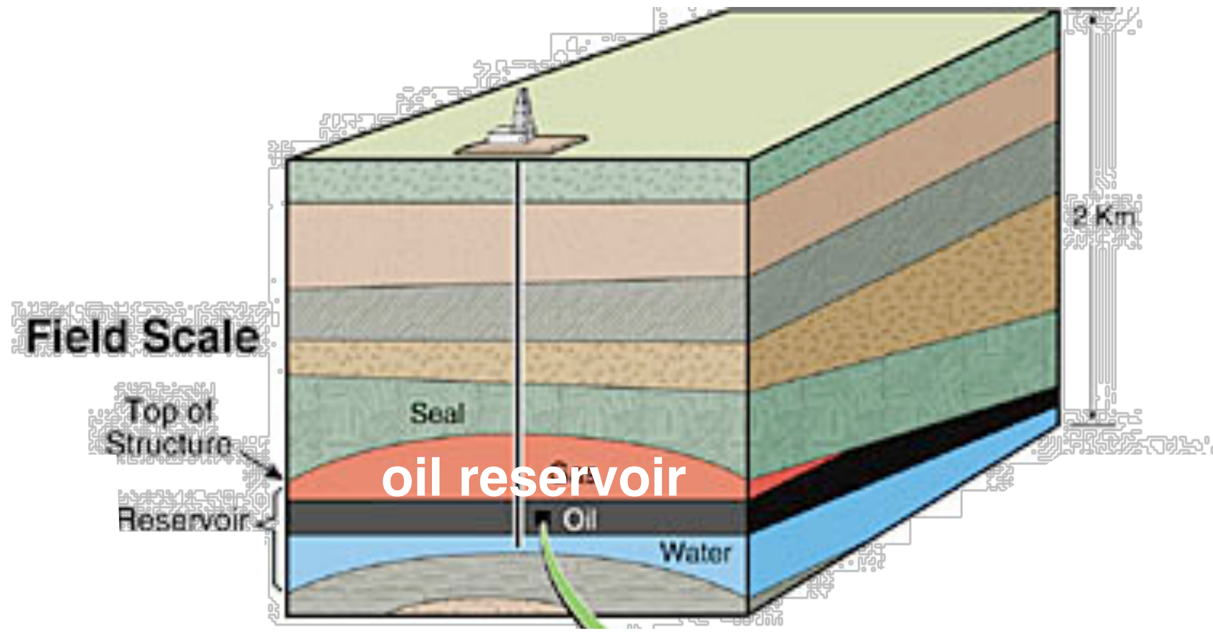

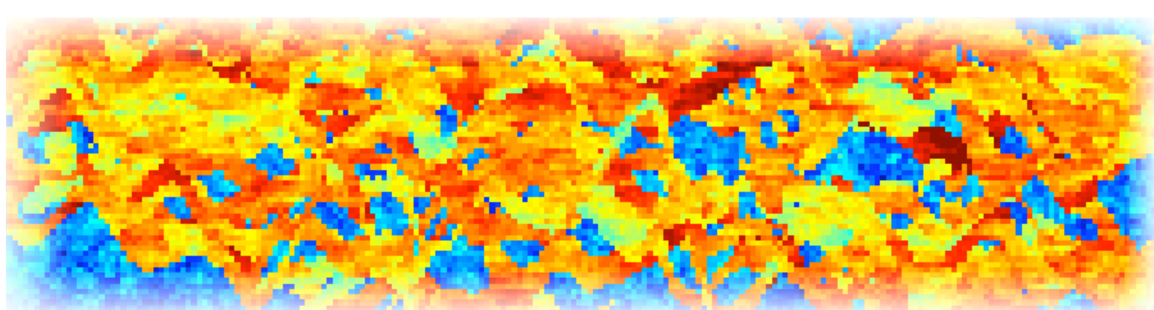

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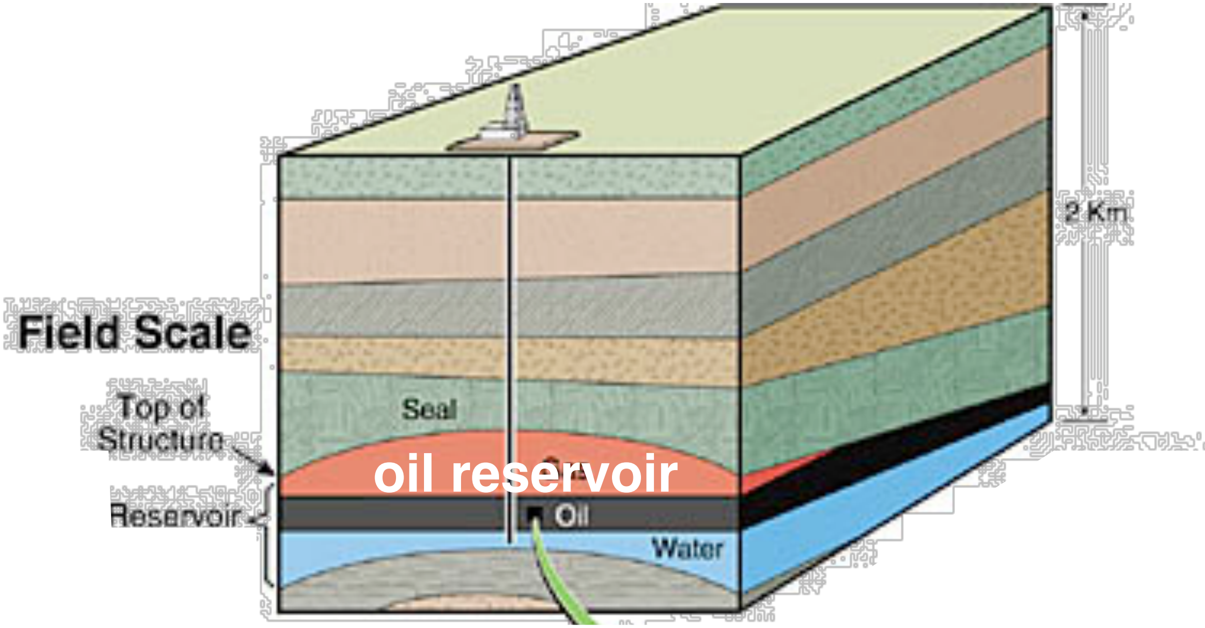
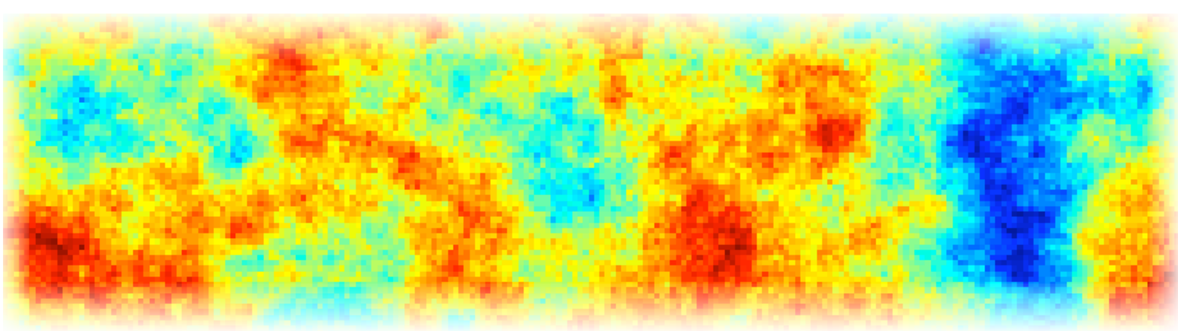
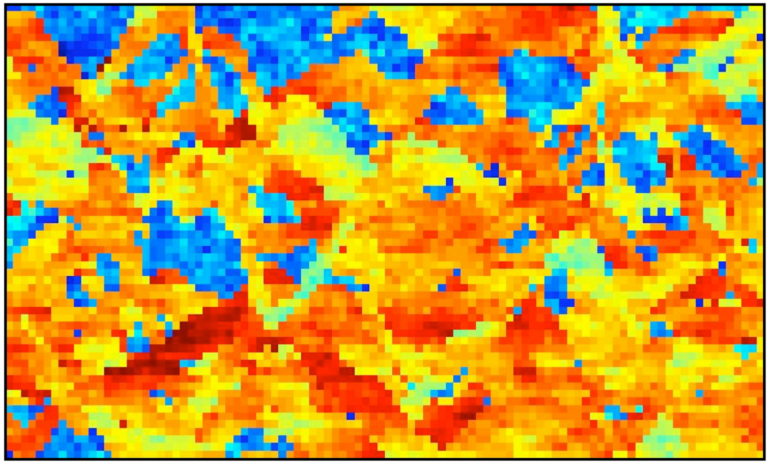
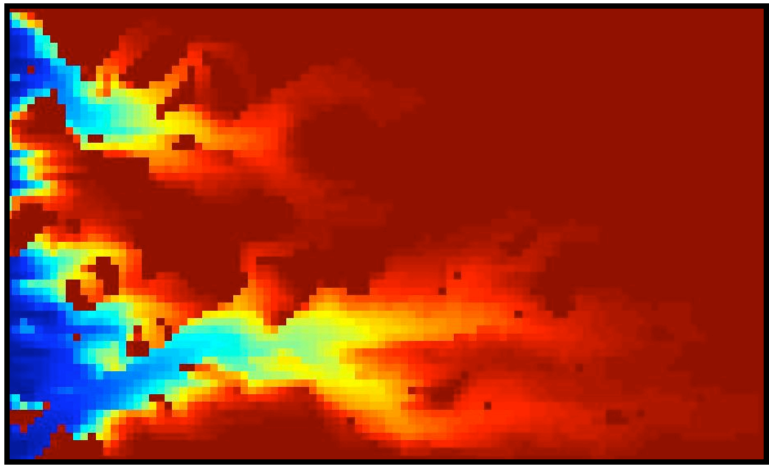




Upscaling vs. Multi-Scale Modeling





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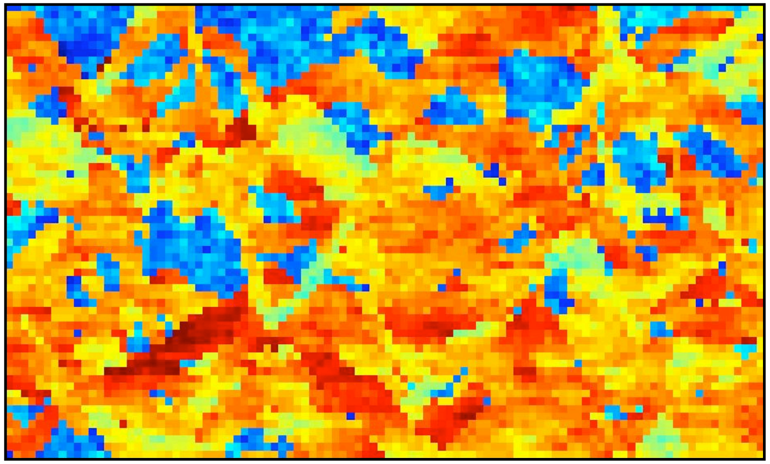

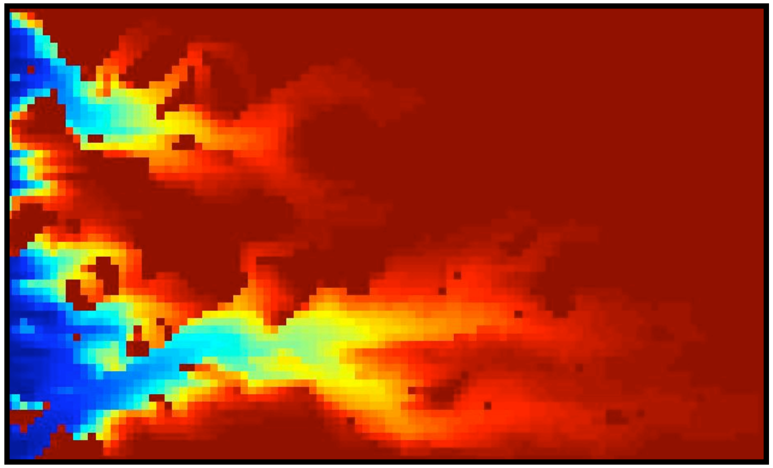

upscaling:

describes large-scale effects of small-scale heterogeneities

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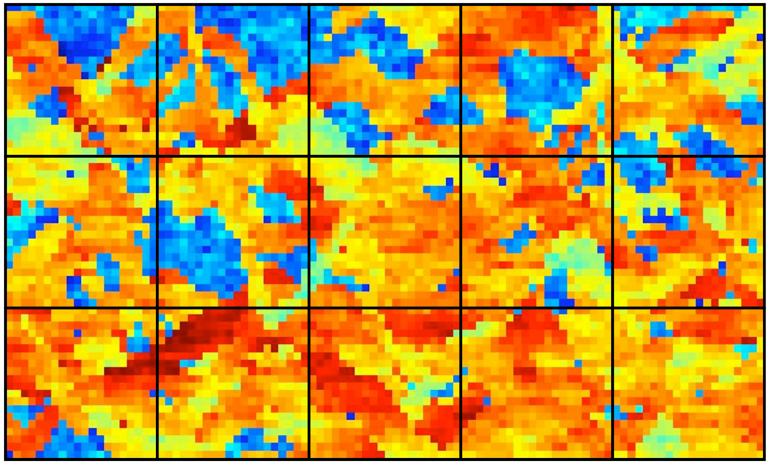

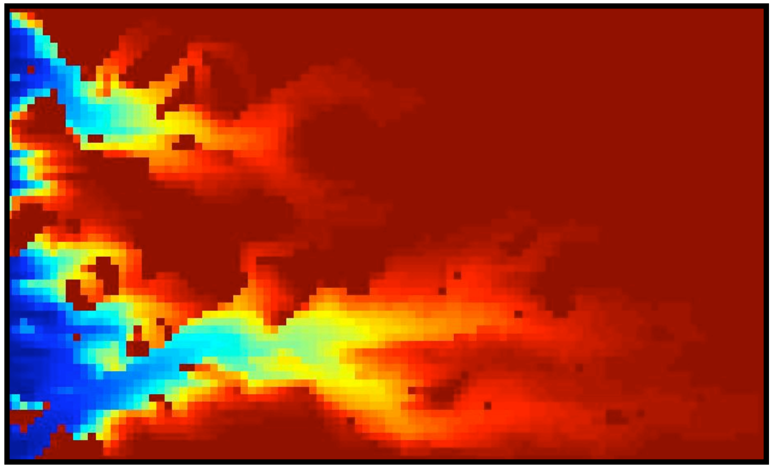

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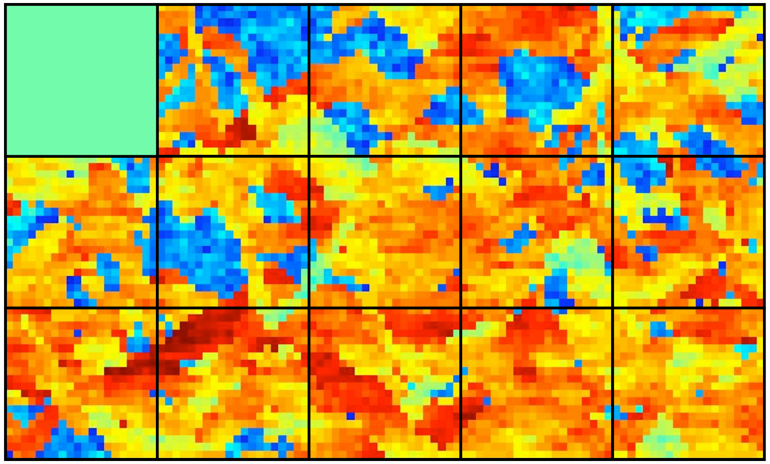

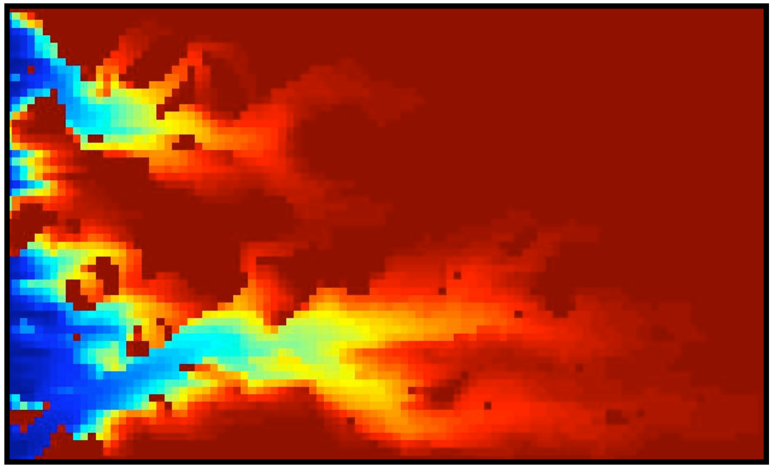

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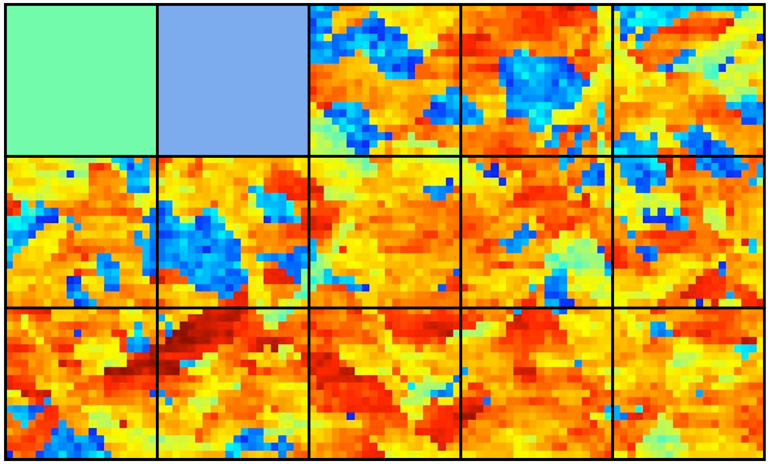

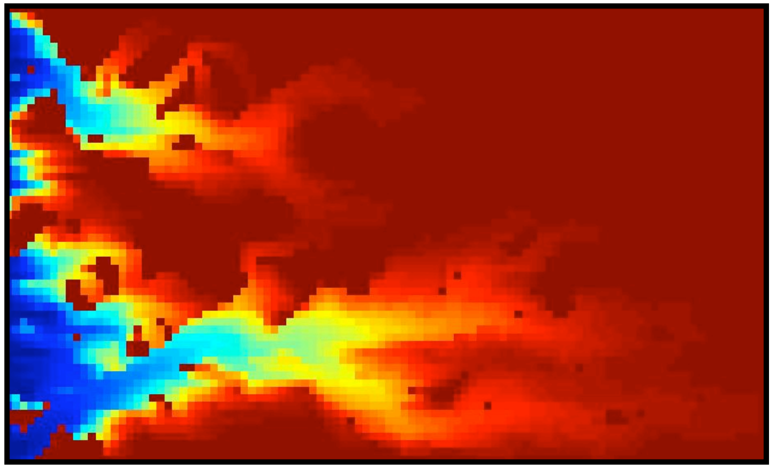

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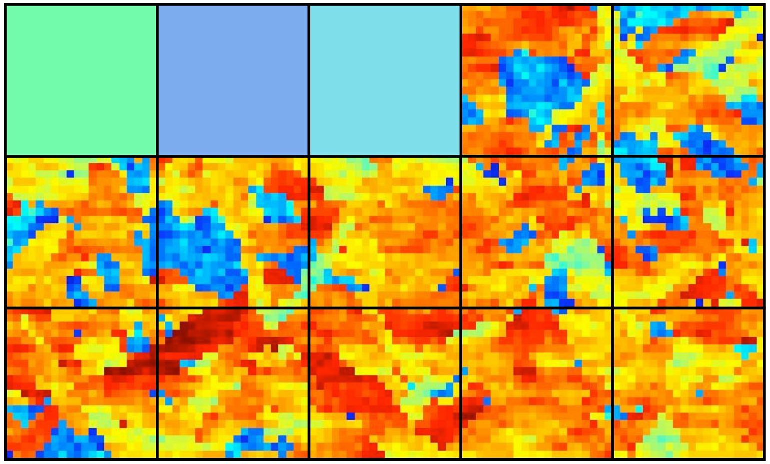

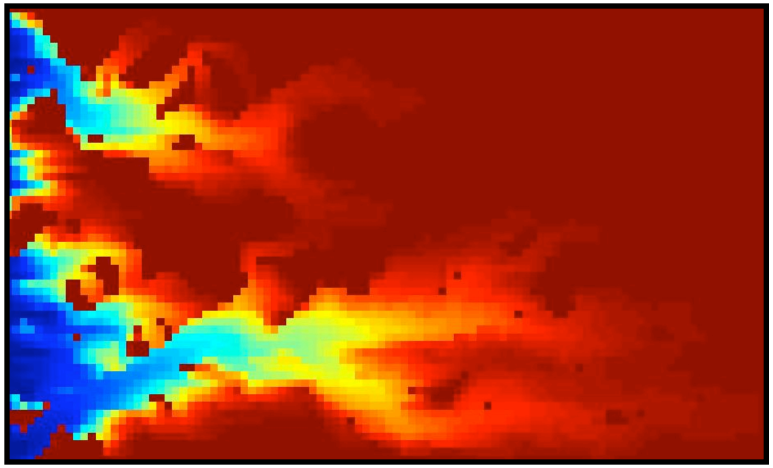

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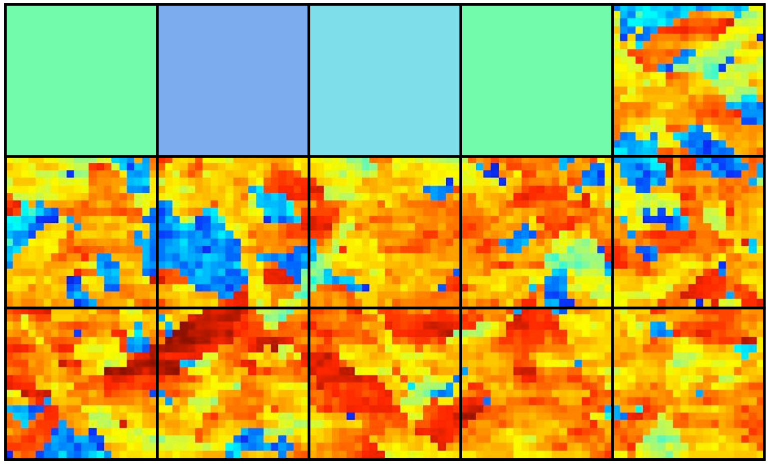

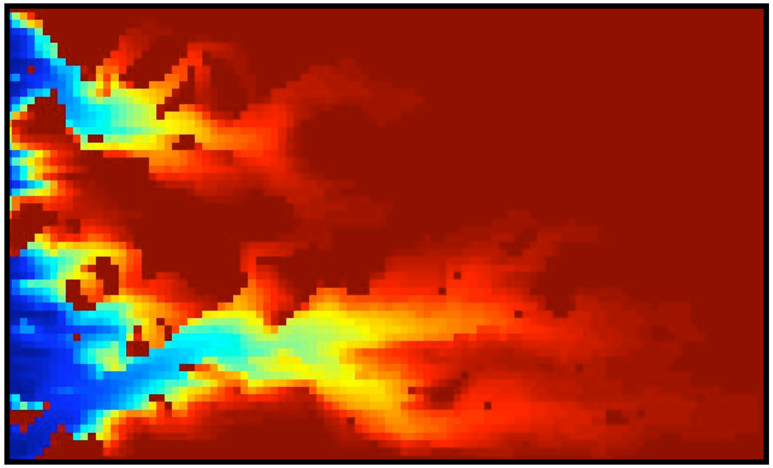

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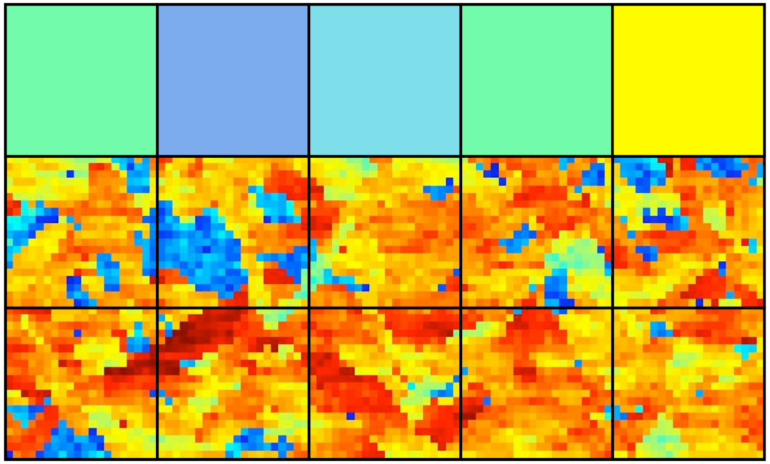

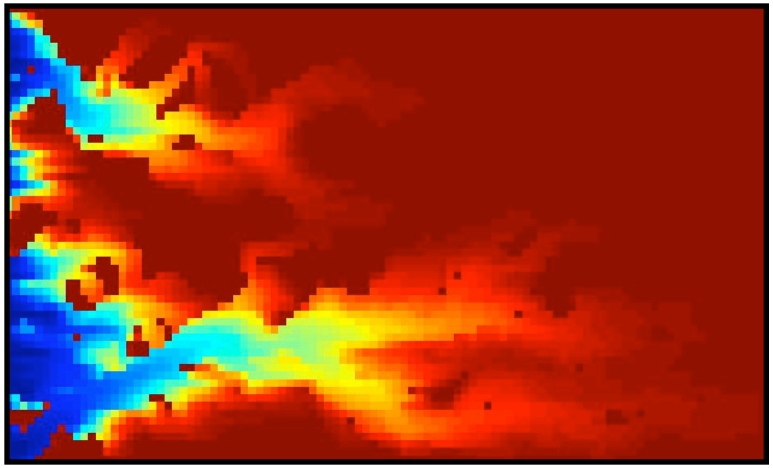

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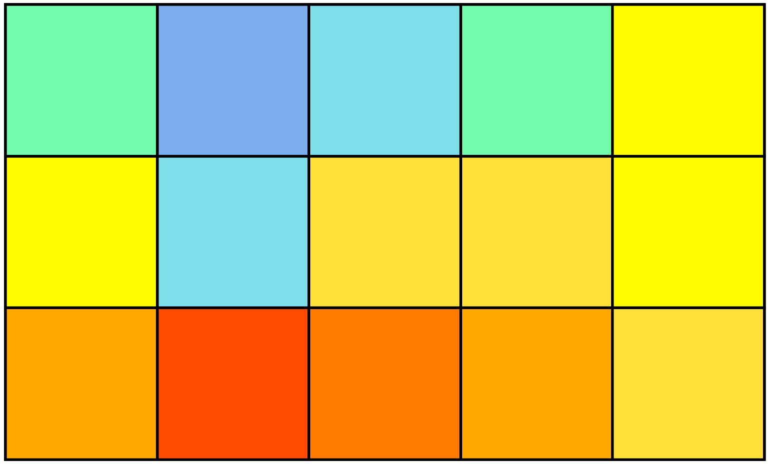

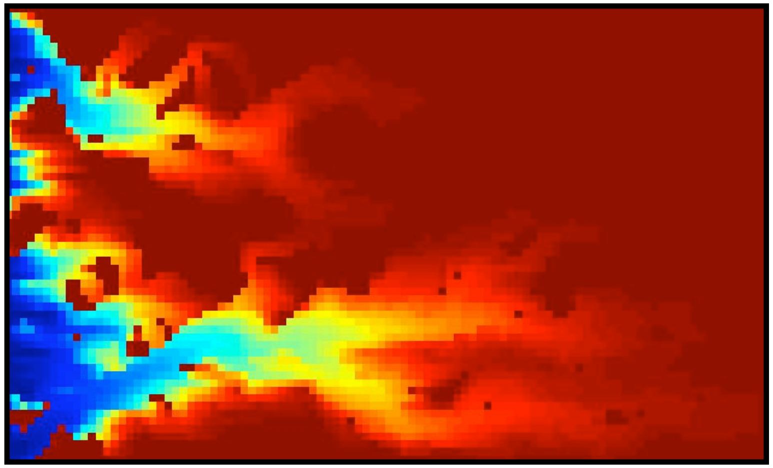

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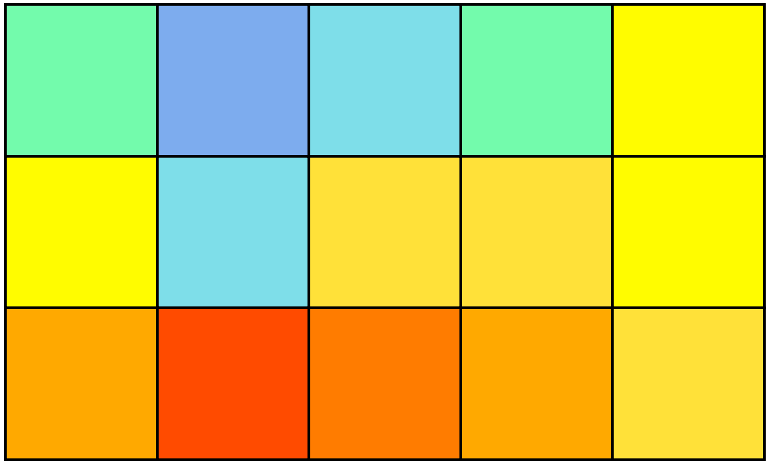

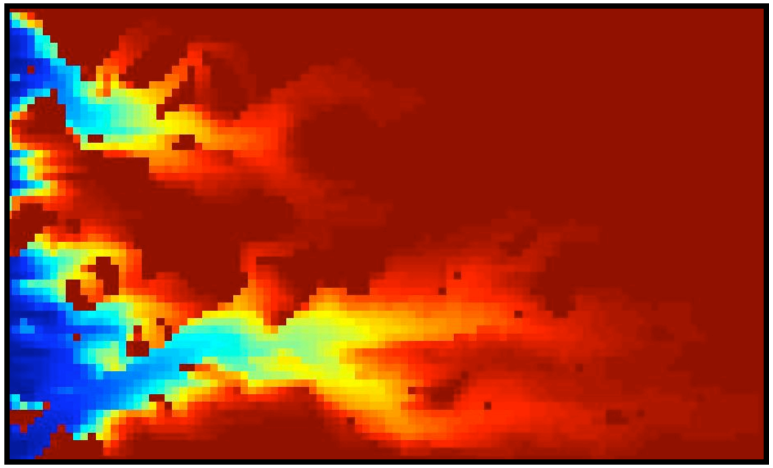

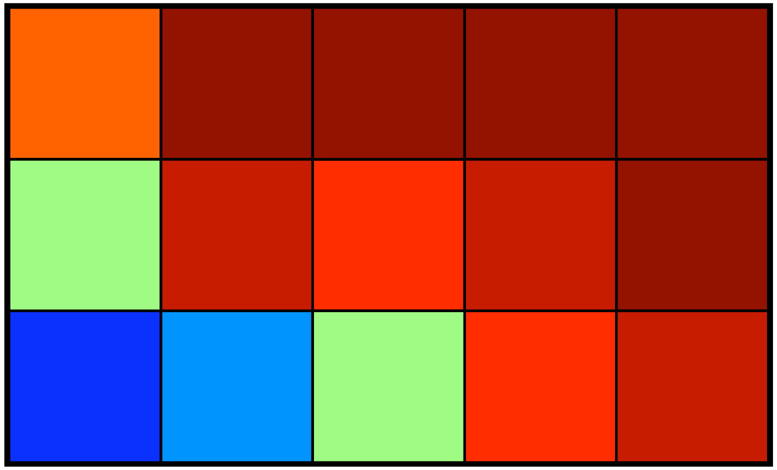

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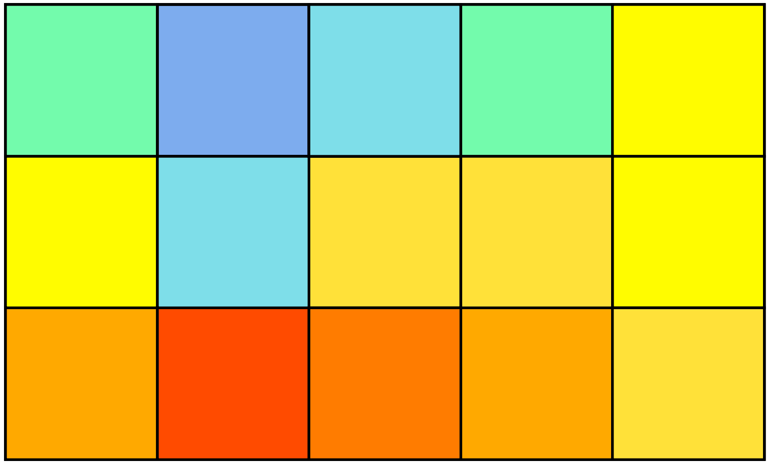

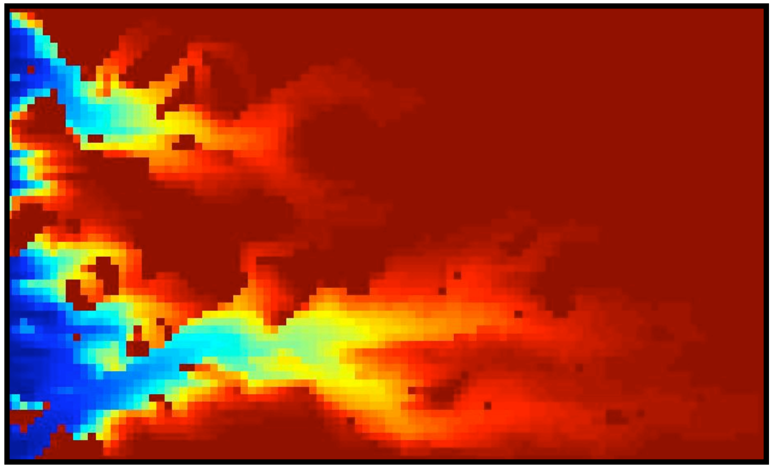

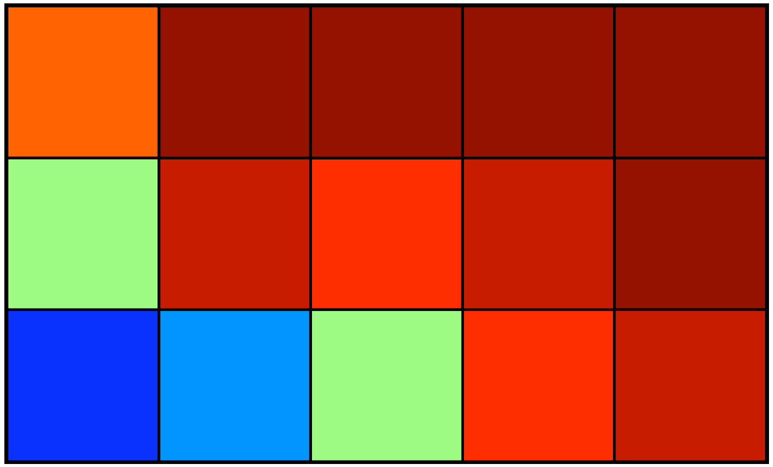

multi-scale methods:

target the solution of the full problem with original resolution

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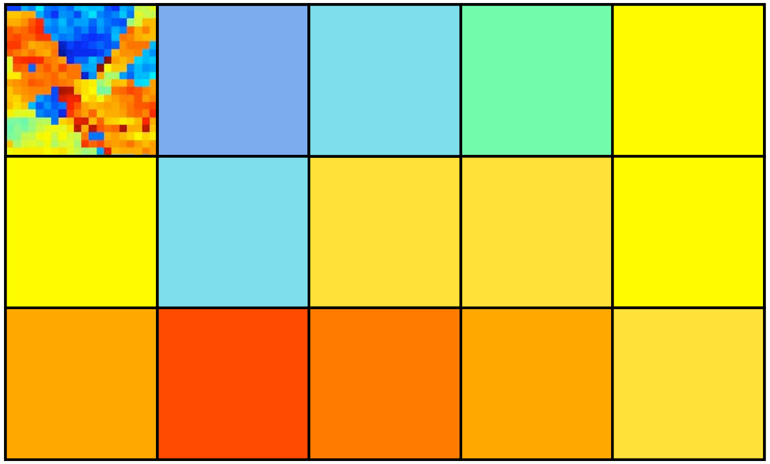

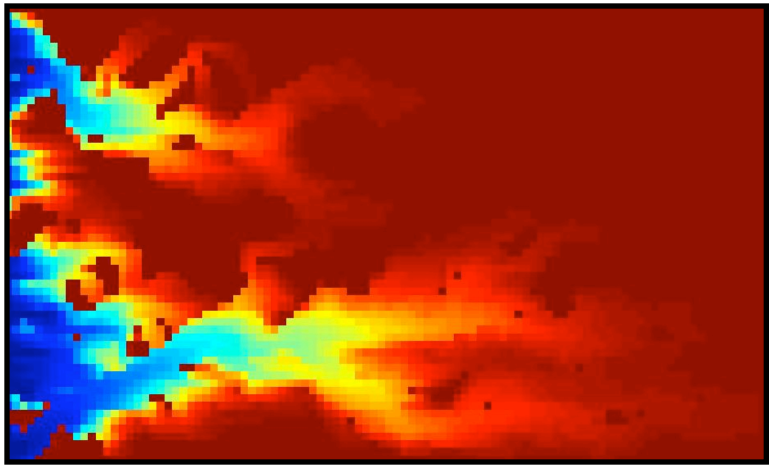

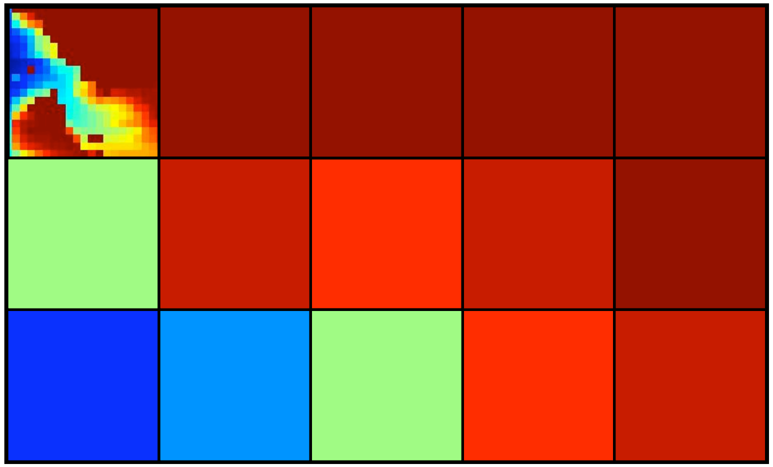

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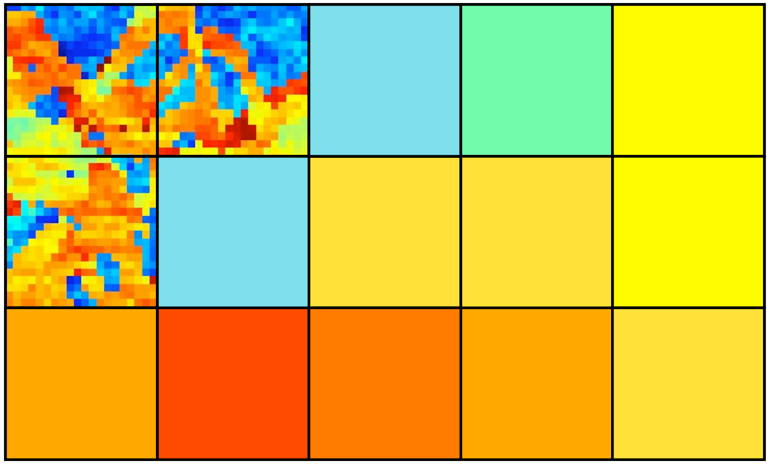

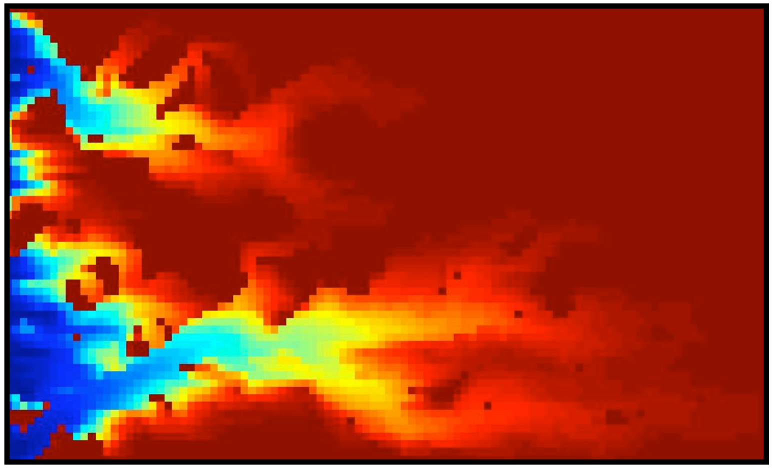

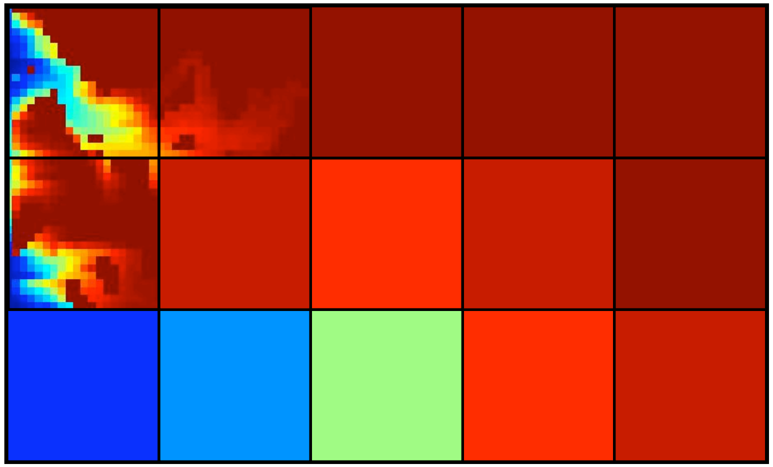

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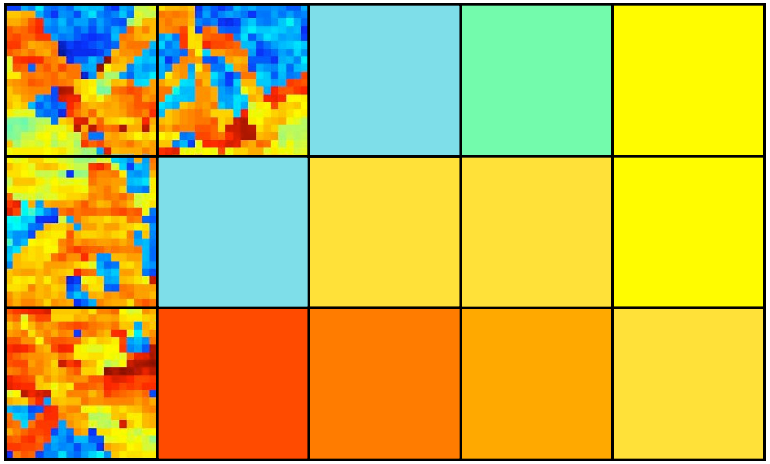

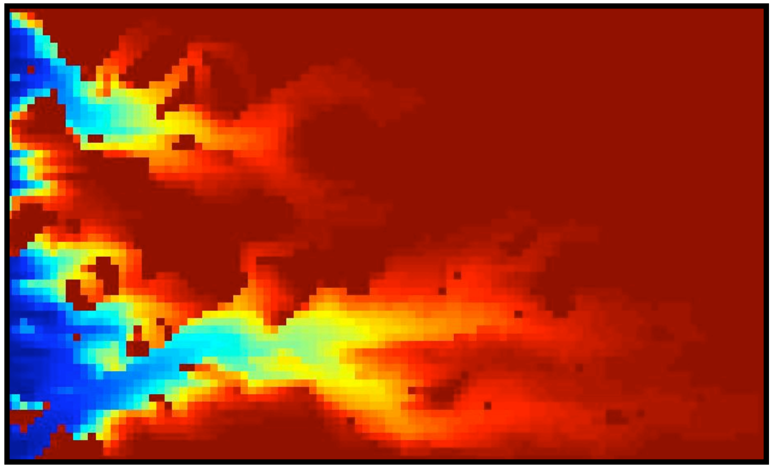

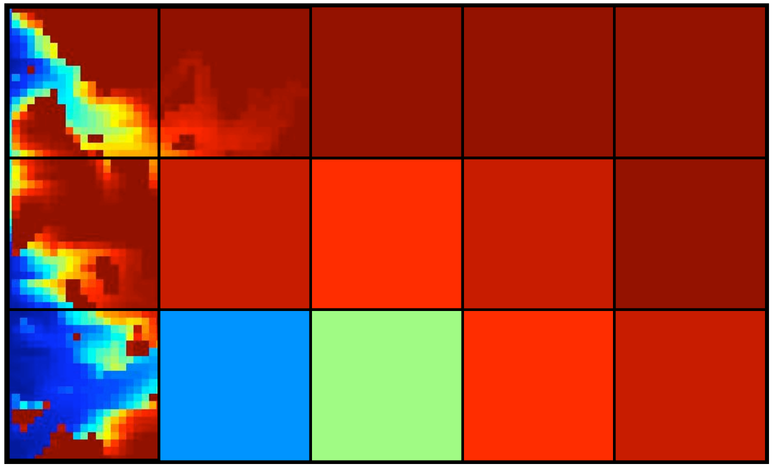

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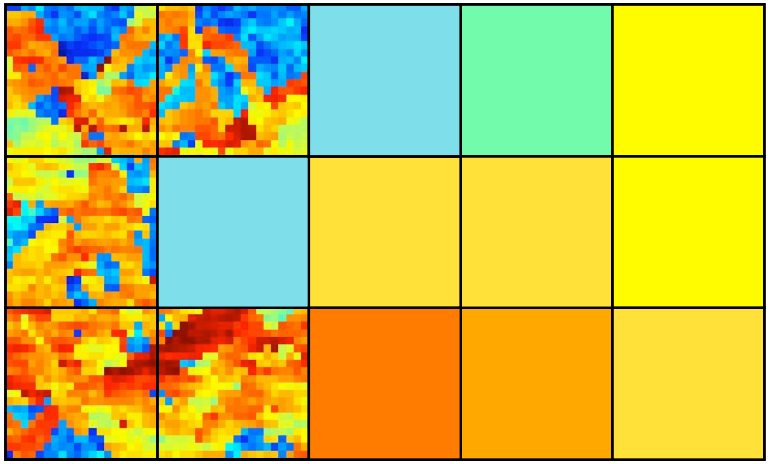

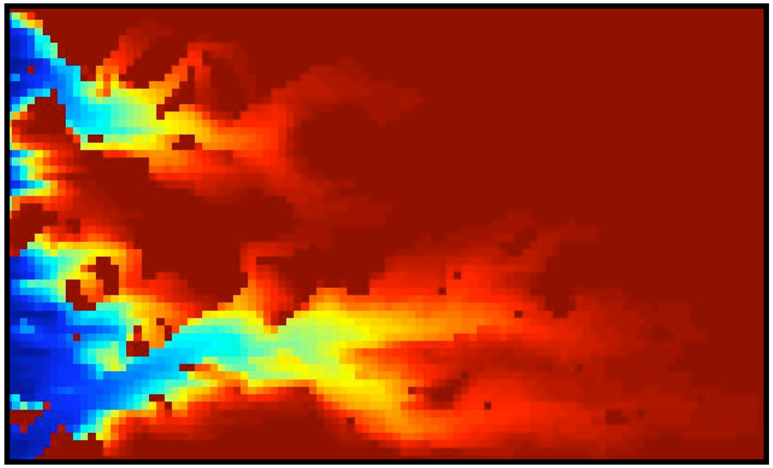

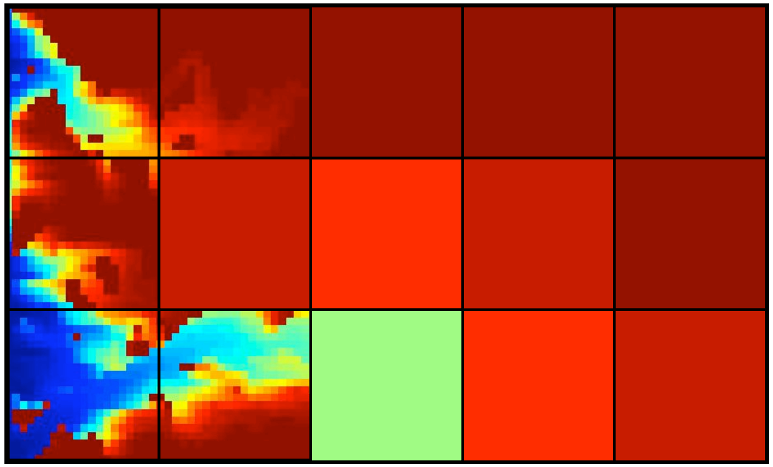

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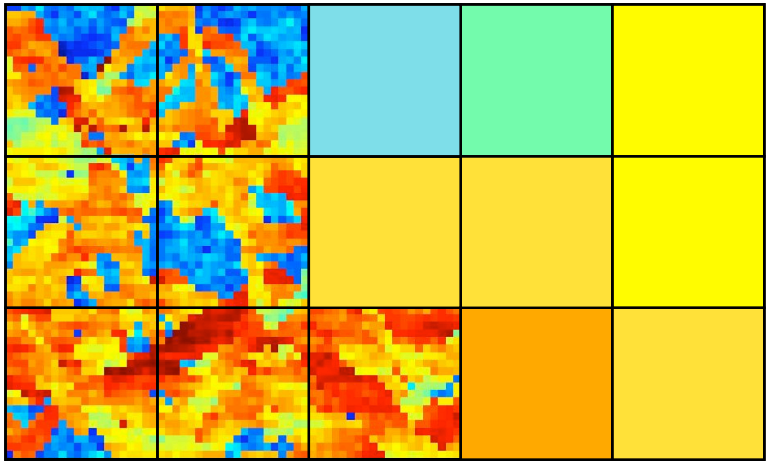

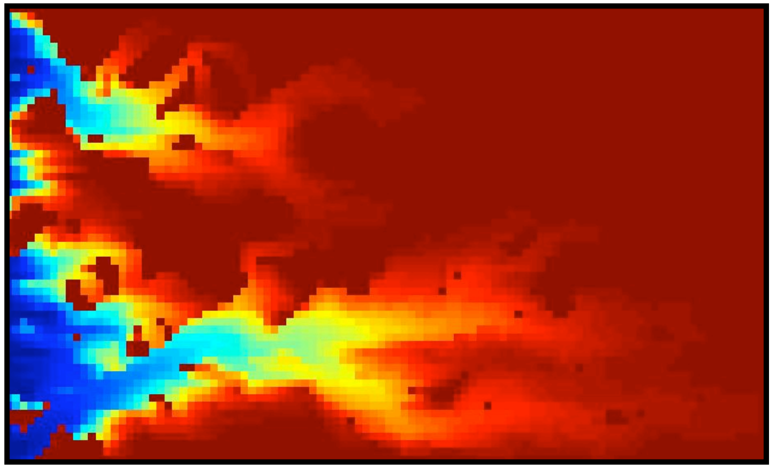

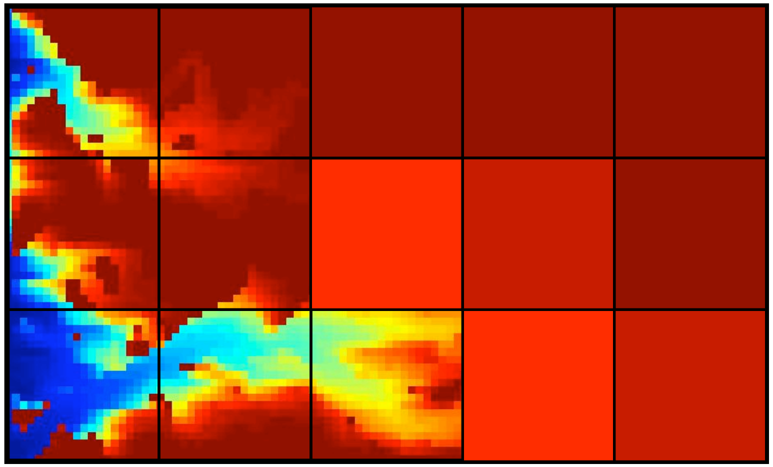

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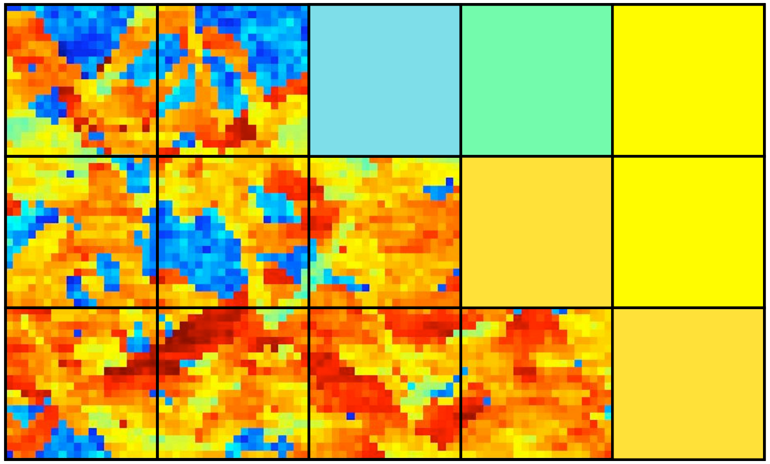

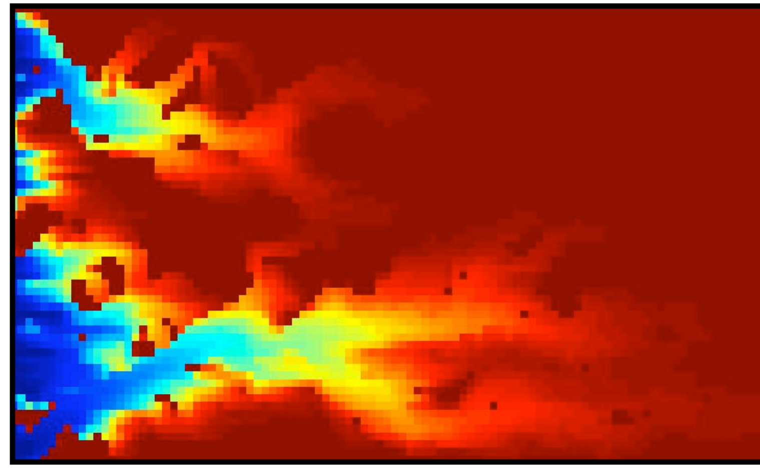

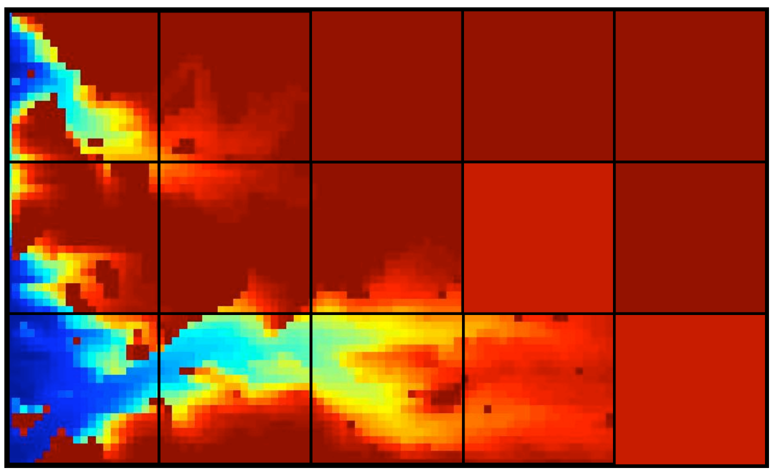

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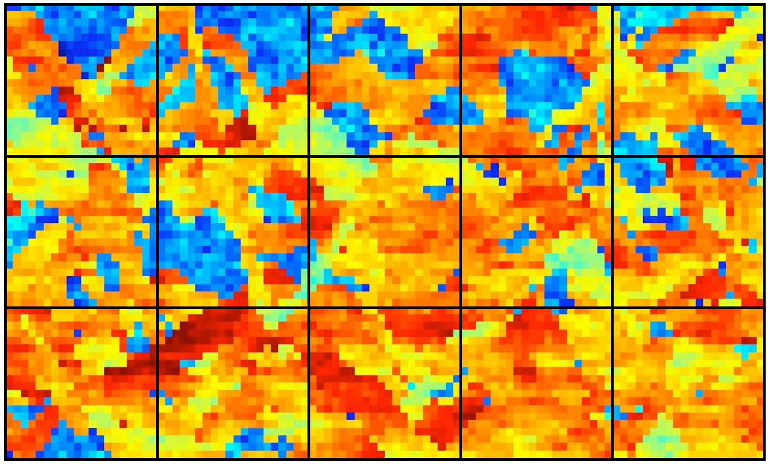

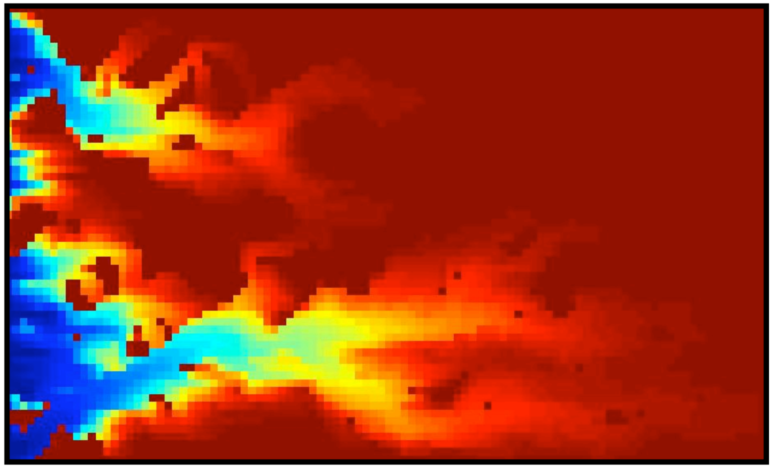

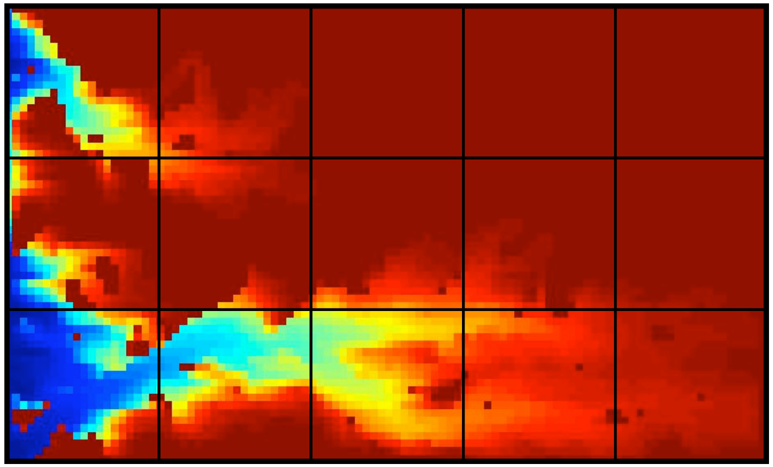

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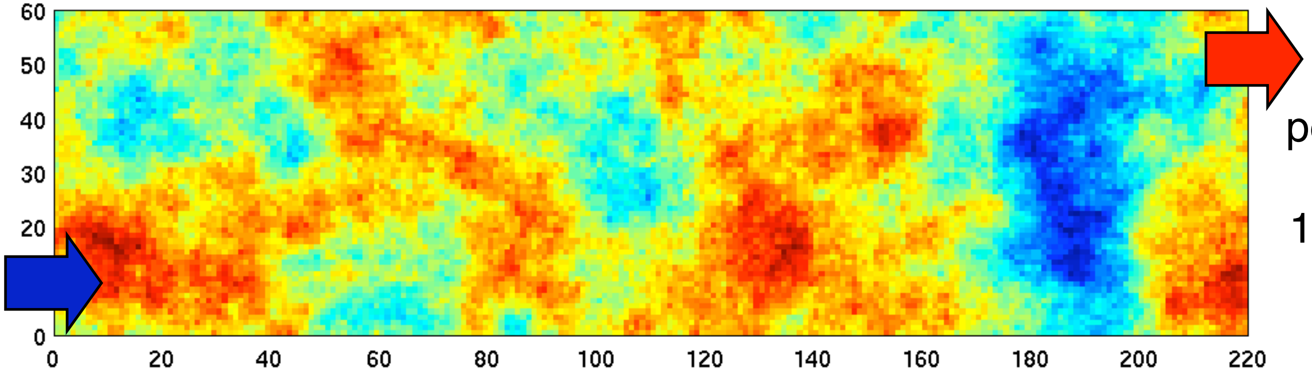
Goal: Reduction of Degrees of Freedom



permeability

13‘020 cells

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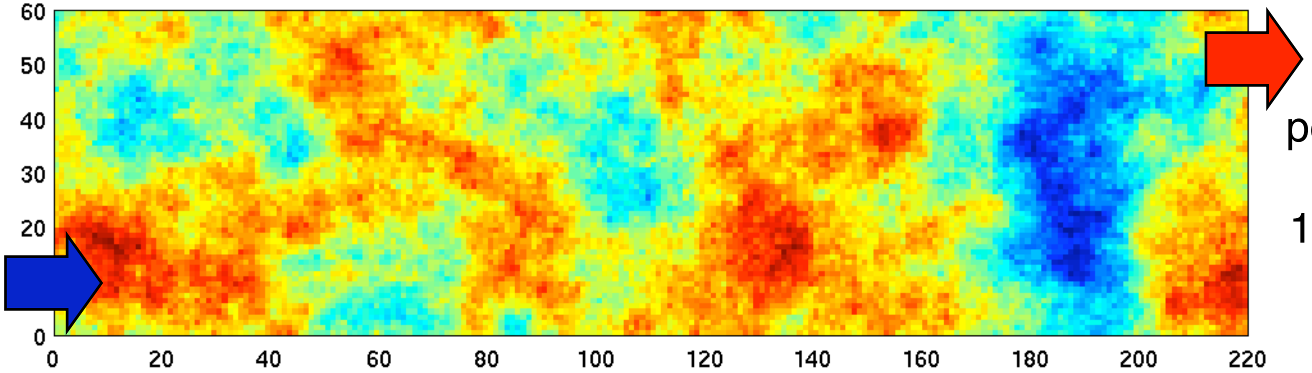

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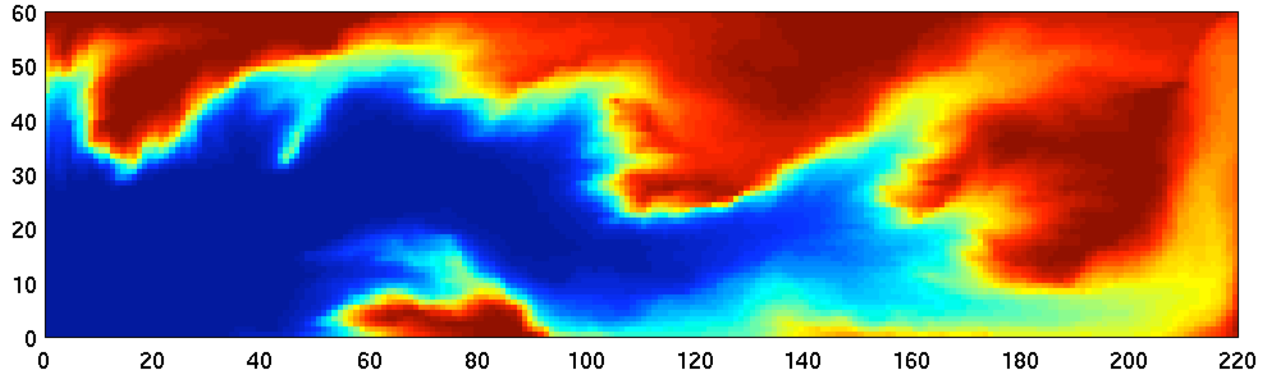

reference

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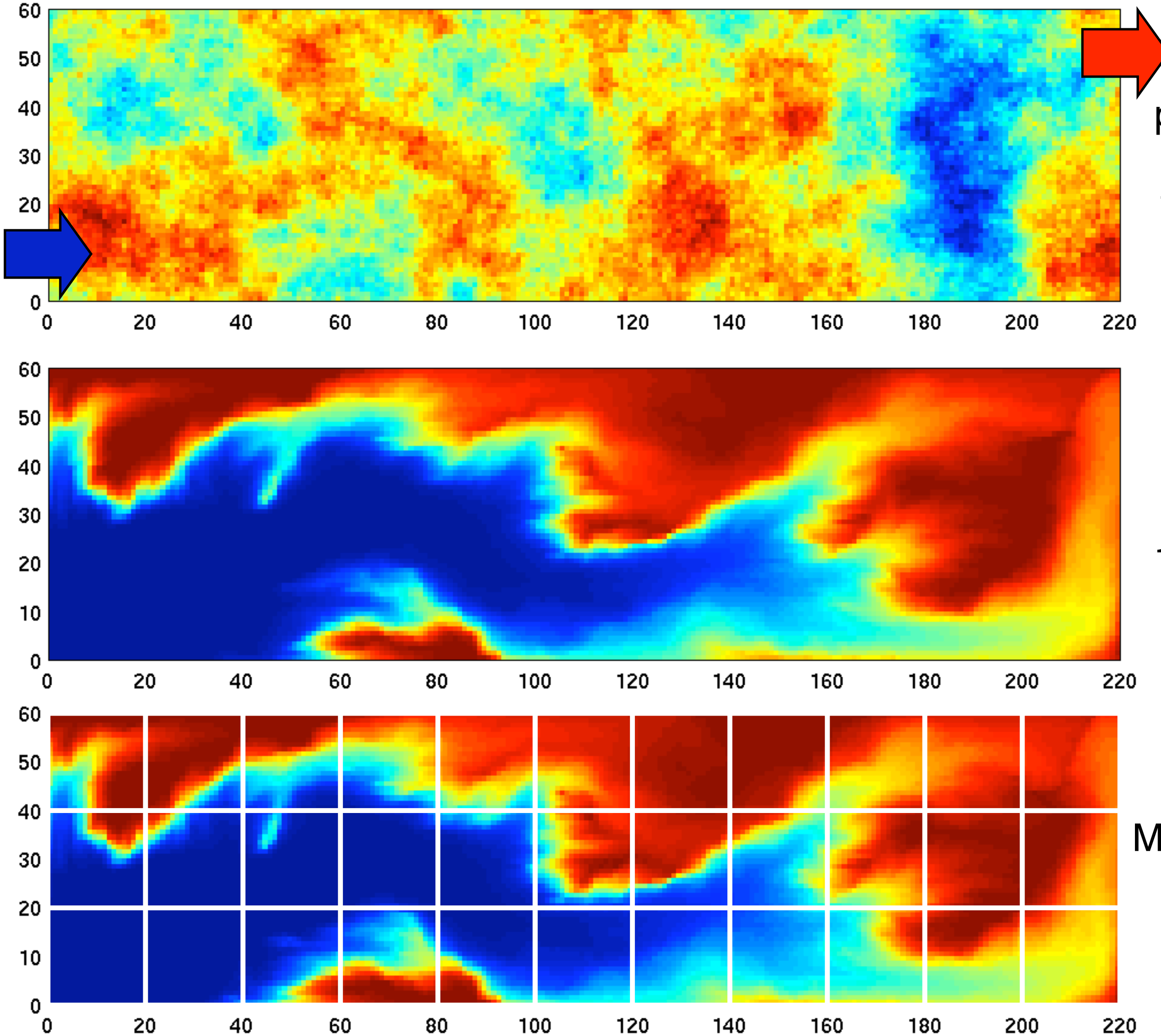

permeability

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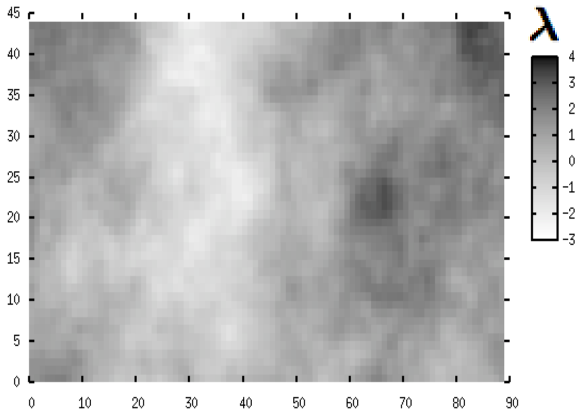
MSFV method
33 cells

Brief Review of MS Methods for Elliptic Problems

- MSFE: multi-scale finite-element method
  (Hou, Wu, Efendiev)
- MMSFE: mixed multi-scale finite-element method
  (Arbogast, Chen, Hou, Aarnes)
- MSFV: multi-scale finite-volume method
  (Jenny, Tchelepi, Lee, Lunati, Wolfsteiner)
- Iterative upscaling with velocity reconstruction
  (Blunt, Durlofsky)
- ...

General Approach

$$\text{solve} \quad \boxed{\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q} \quad \text{on } \Omega$$



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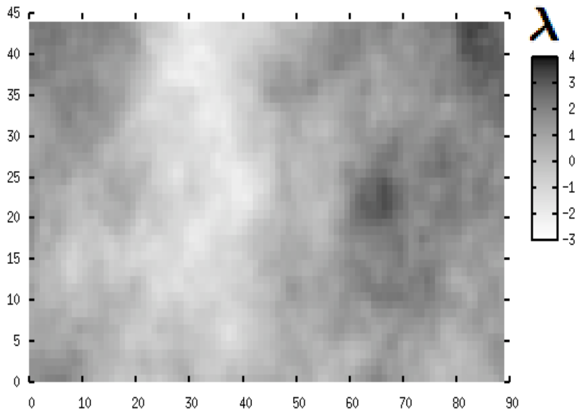

use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{k=1}^{M}\left[\bar{p}_k \Phi_k\right] + \Phi$$

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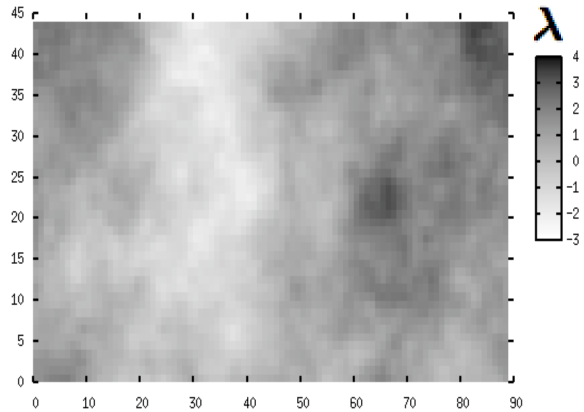
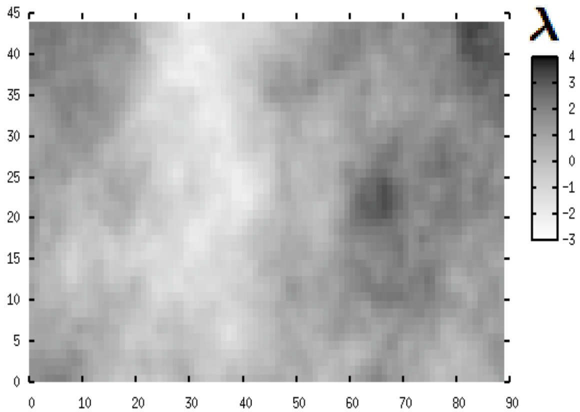


use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{k=1}^{M}[\bar{p}_k\Phi_k] + \Phi$$

$$\text{if } \alpha = 0 \Rightarrow \quad \sum_{k=1}^{M}[\Phi_k] \equiv 1$$

General Approach

solve $$\boxed{\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q}$$ on Ω

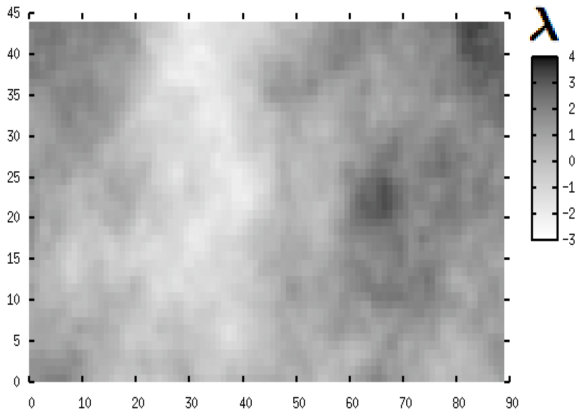


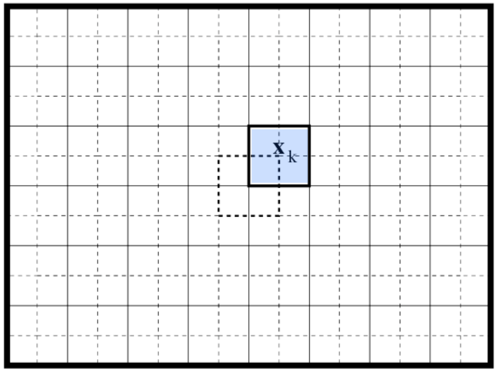
use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{k=1}^{M} [\bar{p}_k \Phi_k] + \Phi$$

solve for $$\int_{\Omega} \Psi_k L(p') dV = \int_{\Omega} \Psi_k q \; dV$$

Multi-Scale Finite Volume (MSFV) Method

solve $$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$ on Ω



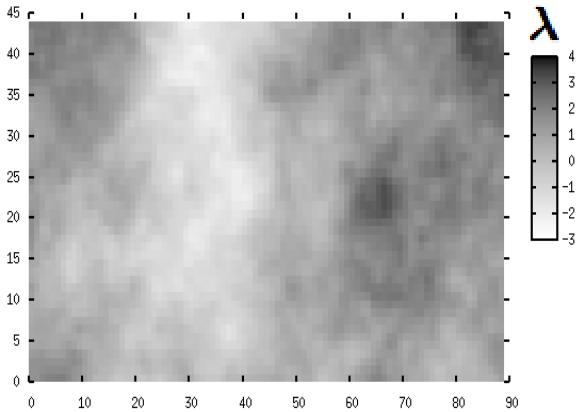


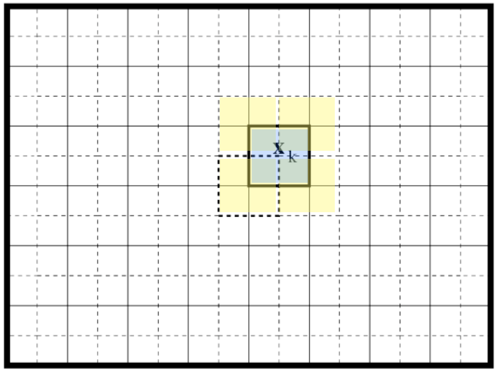
use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{k=1}^{M}\left[\bar{p}_k \Phi_k\right] + \Phi$$

solve for $$\int_\Omega \Psi_k L(p') dV = \int_\Omega \Psi_k q \; dV$$

Multi-Scale Finite Volume (MSFV) Method

solve $$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$ on Ω



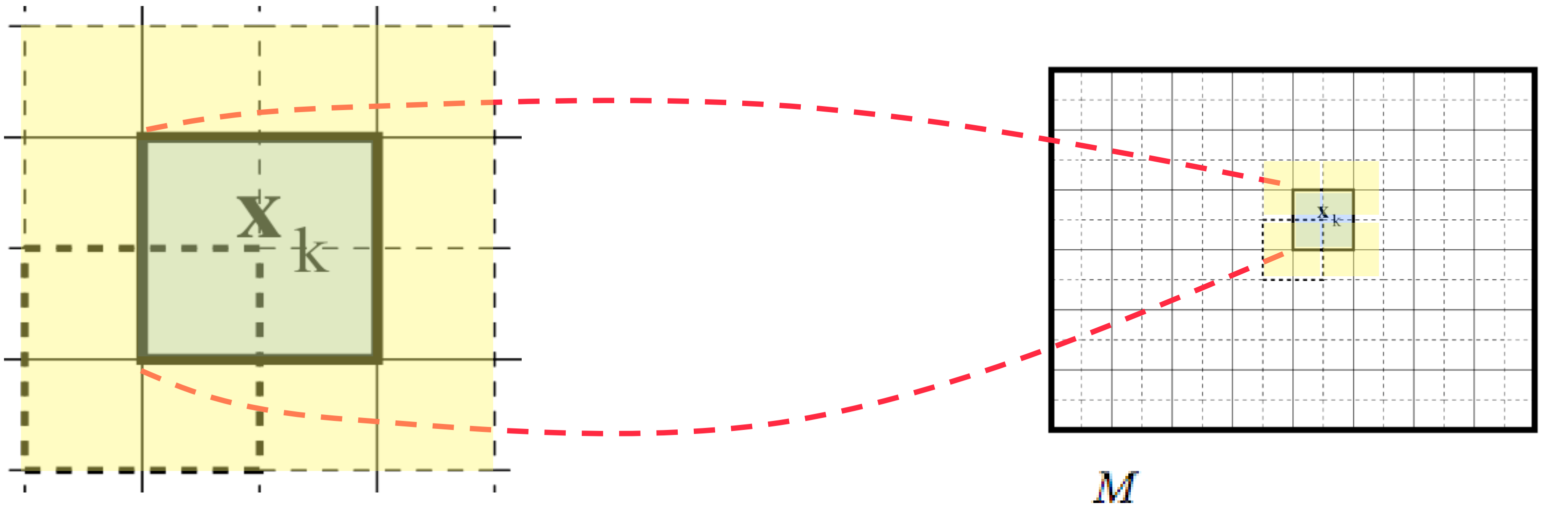


use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{k=1}^{M}\left[\bar{p}_k \Phi_k\right] + \Phi$$

solve for $$\int_\Omega \Psi_k L(p')dV = \int_\Omega \Psi_k q \; dV$$

Multi-Scale Finite Volume (MSFV) Method

solve $$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$ on Ω

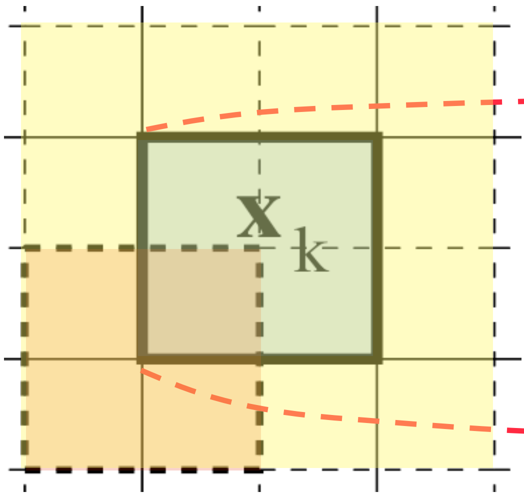


use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{k=1}^{M} [\bar{p}_k \Phi_k] + \Phi$$

solve for $$\int_\Omega \Psi_k L(p') dV = \int_\Omega \Psi_k q \; dV$$

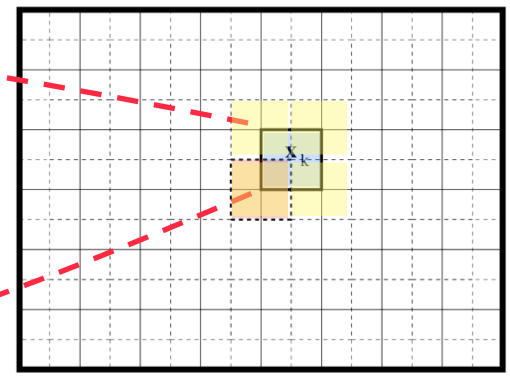
Multi-Scale Finite Volume (MSFV) Method

solve $$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$ on Ω



$$\Phi_k = \sum_{h=1}^{N}\left[\Phi_k^h\right]$$

$$\Phi = \sum_{h=1}^{N}\Phi^h$$

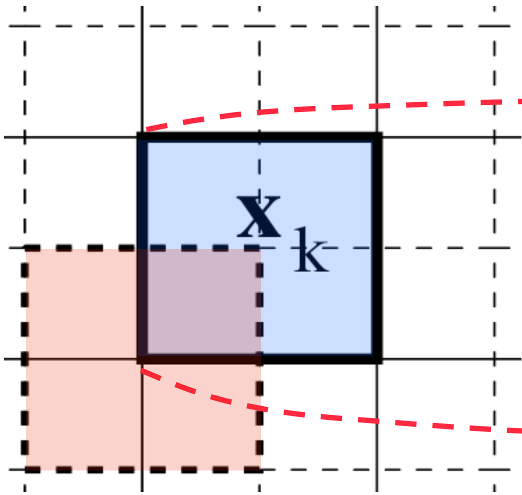


use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{k=1}^{M}\left[\bar{p}_k\Phi_k\right] + \Phi$$

solve for $$\int_\Omega \Psi_k L(p')dV = \int_\Omega \Psi_k q\ dV$$

Multi-Scale Finite Volume (MSFV) Method

solve $$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$ on Ω



$$\Phi_k = \sum_{h=1}^{N}\left[\Phi_k^h\right]$$

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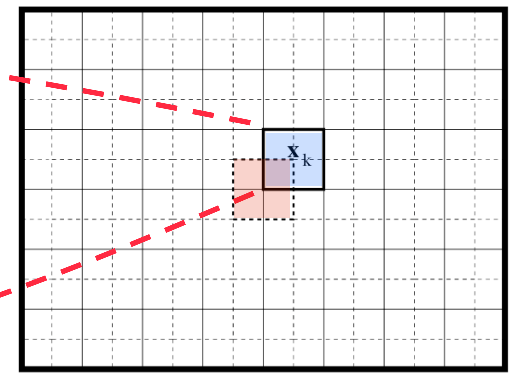


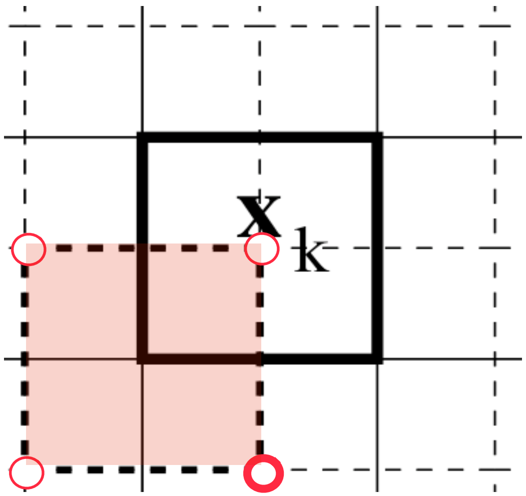
use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{h=1}^{N}\left\{\sum_{k=1}^{M}\left[\Phi_k^h \bar{p}_k\right] + \Phi^h\right\}$$

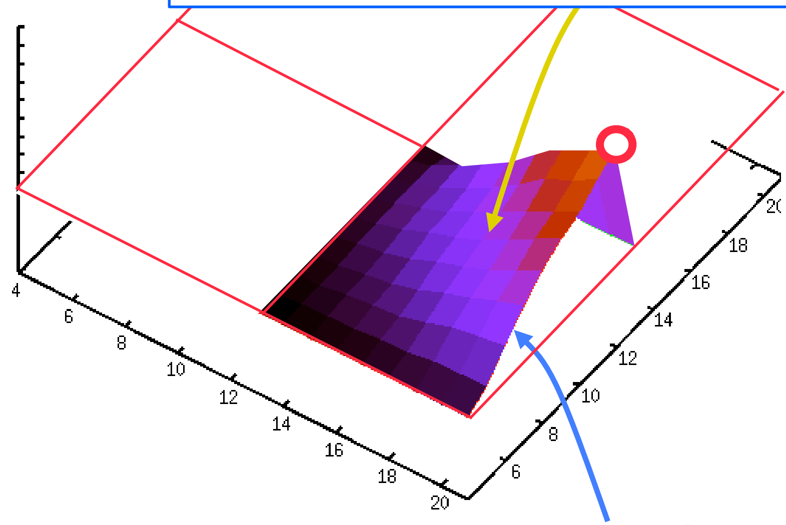
solve for $$\int_\Omega \Psi_k L(p')dV = \int_\Omega \Psi_k q\ dV$$

Basis

$$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$

$$\alpha \Phi_k^h + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial \Phi_k^h}{\partial x_j}\right\} = 0$$





$$\frac{\partial}{\partial x_n}\left\{\lambda_{ij}\frac{\partial \Phi_k^h}{\partial x_j}\tilde{\nu}_i^h\right\}\tilde{\nu}_n^h = 0$$

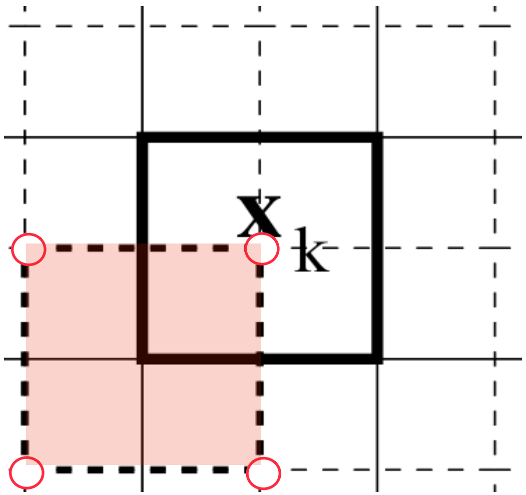
use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{h=1}^{N}\left\{\sum_{k=1}^{M}\left[\Phi_k^h \bar{p}_k\right] + \Phi^h\right\}$$

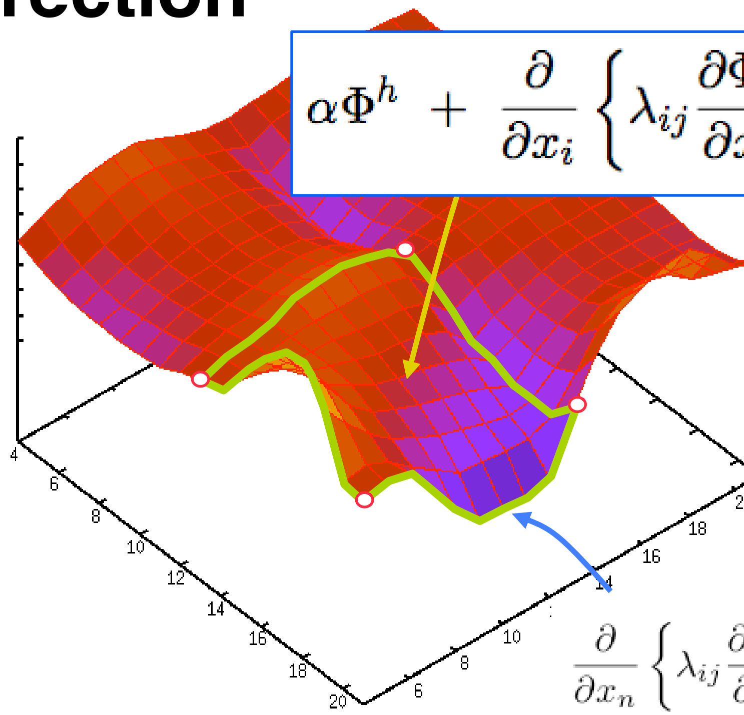
solve for $$\int_\Omega \Psi_k L(p')dV = \int_\Omega \Psi_k q\ dV$$

Correction

$$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$

$$\alpha\Phi^h + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial \Phi^h}{\partial x_j}\right\} = q$$





$$\frac{\partial}{\partial x_n}\left\{\lambda_{ij}\frac{\partial \Phi^h}{\partial x_j}\tilde{\nu}_i^h\right\}\tilde{\nu}_n^h = 0$$

use the approximation

$$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{h=1}^{N}\left\{\sum_{k=1}^{M}\left[\Phi_k^h \bar{p}_k\right] + \Phi^h\right\}$$

solve for

$$\int_\Omega \Psi_k L(p')dV = \int_\Omega \Psi_k q \; dV$$

Coarse System

problem $$\alpha p + \frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$ on Ω

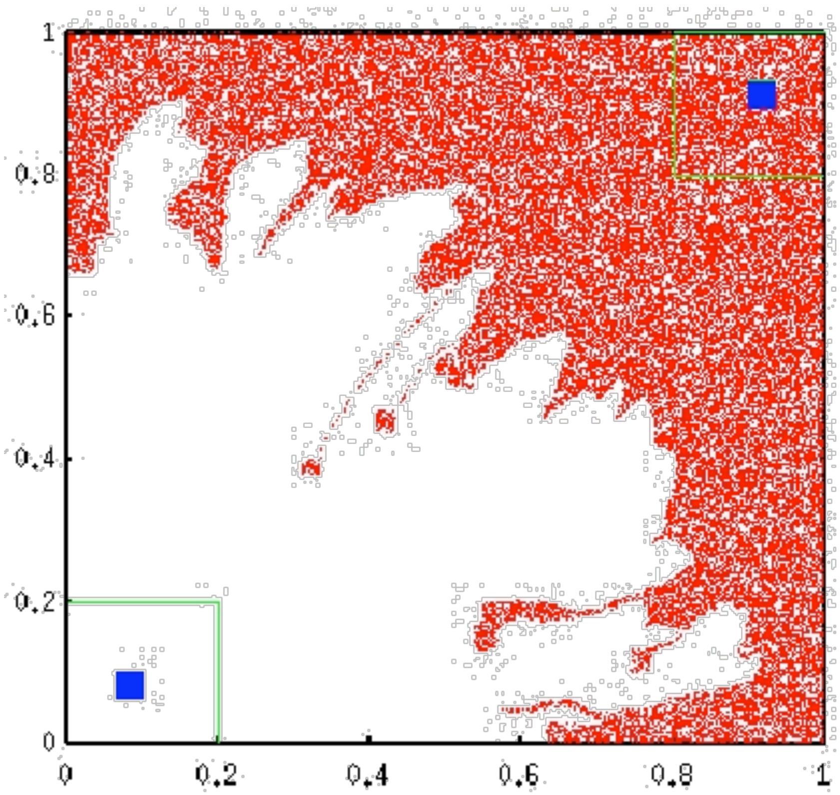
use the approximation $$p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{h=1}^{N}\left\{\sum_{k=1}^{M}\left[\Phi_k^h \bar{p}_k\right] + \Phi^h\right\}$$

coarse system $$\int_\Omega \Psi_k L(p')dV = \int_\Omega \Psi_k q \; dV$$

$$\Rightarrow \boldsymbol{A} \cdot \bar{\boldsymbol{p}} = \boldsymbol{R}$$

Local Conservativity

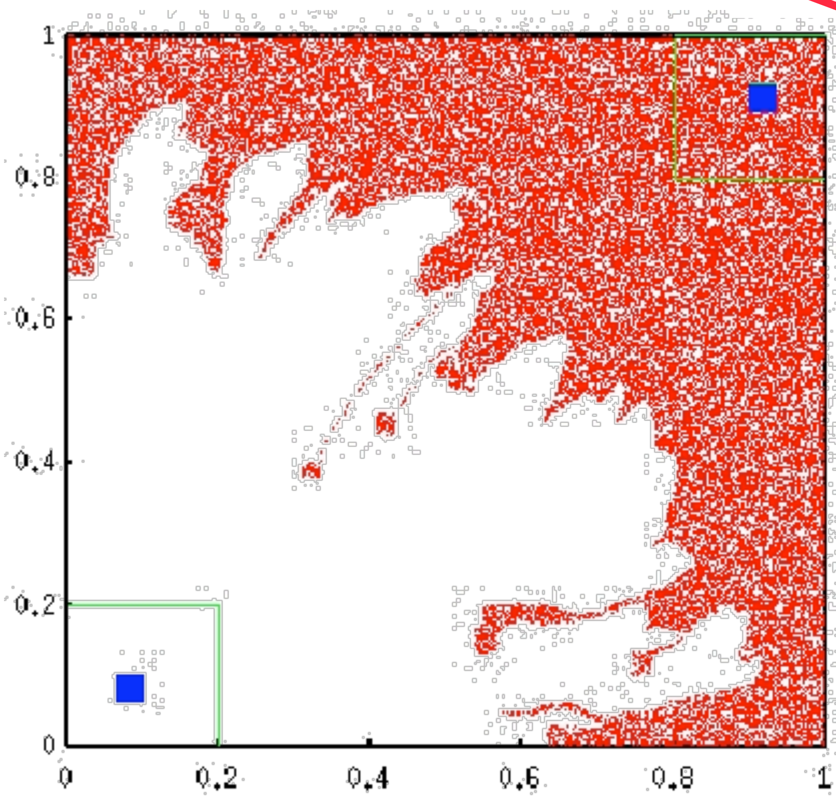
$$\frac{\partial}{\partial x_i}\left\{\lambda_{ij}\frac{\partial p}{\partial x_j}\right\} = L(p) = q$$

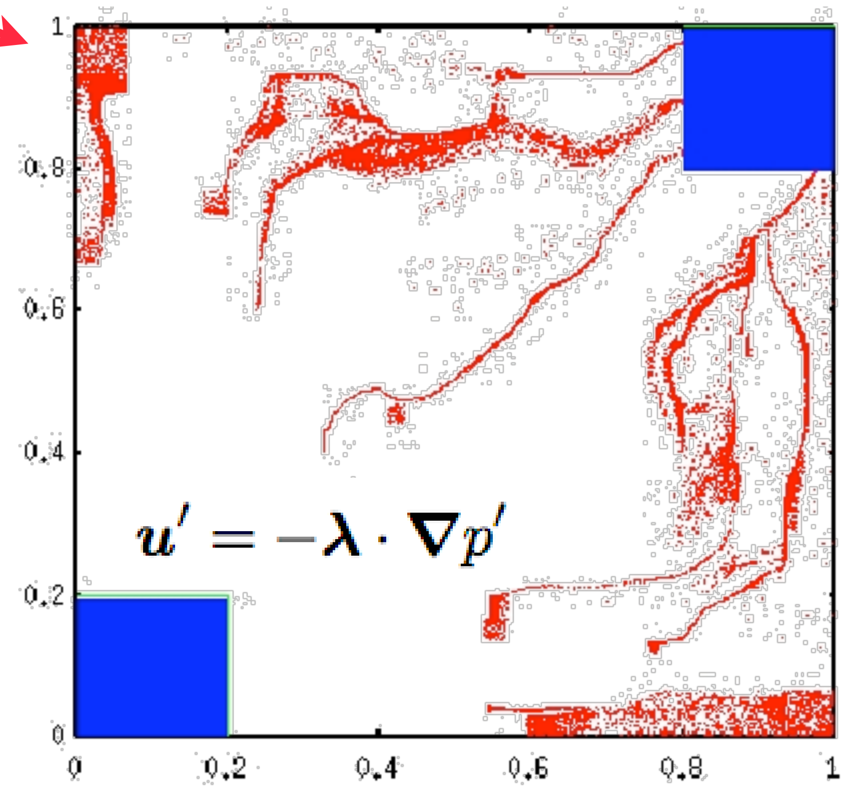


$$\frac{\partial \phi c}{\partial t} + \frac{\partial}{\partial x_i}\left\{\underbrace{-\lambda_{ij}\frac{\partial p}{\partial x_j}}_{u_i} c\right\} = q$$

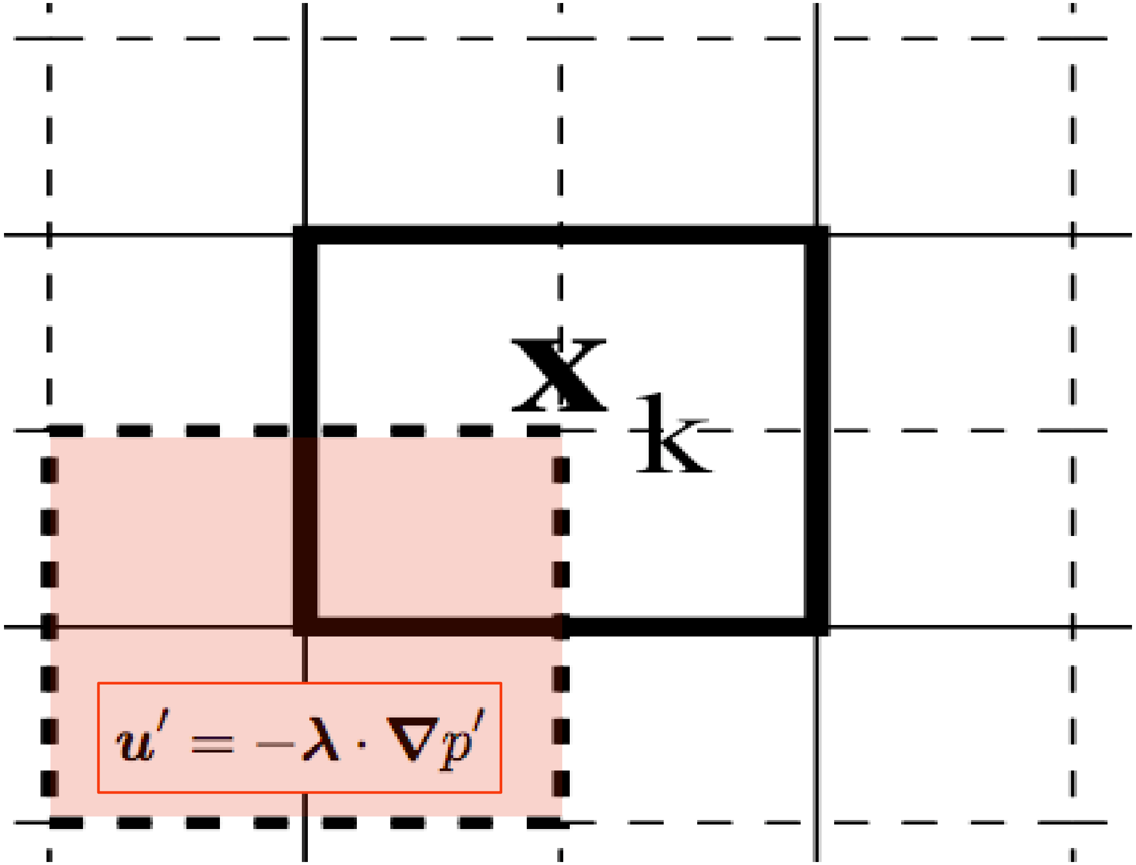
Local Conservativity

with the approximation $p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{h=1}^{N} \left\{ \sum_{k=1}^{M} \left[\Phi_k^h \bar{p}_k \right] + \Phi^h \right\}$

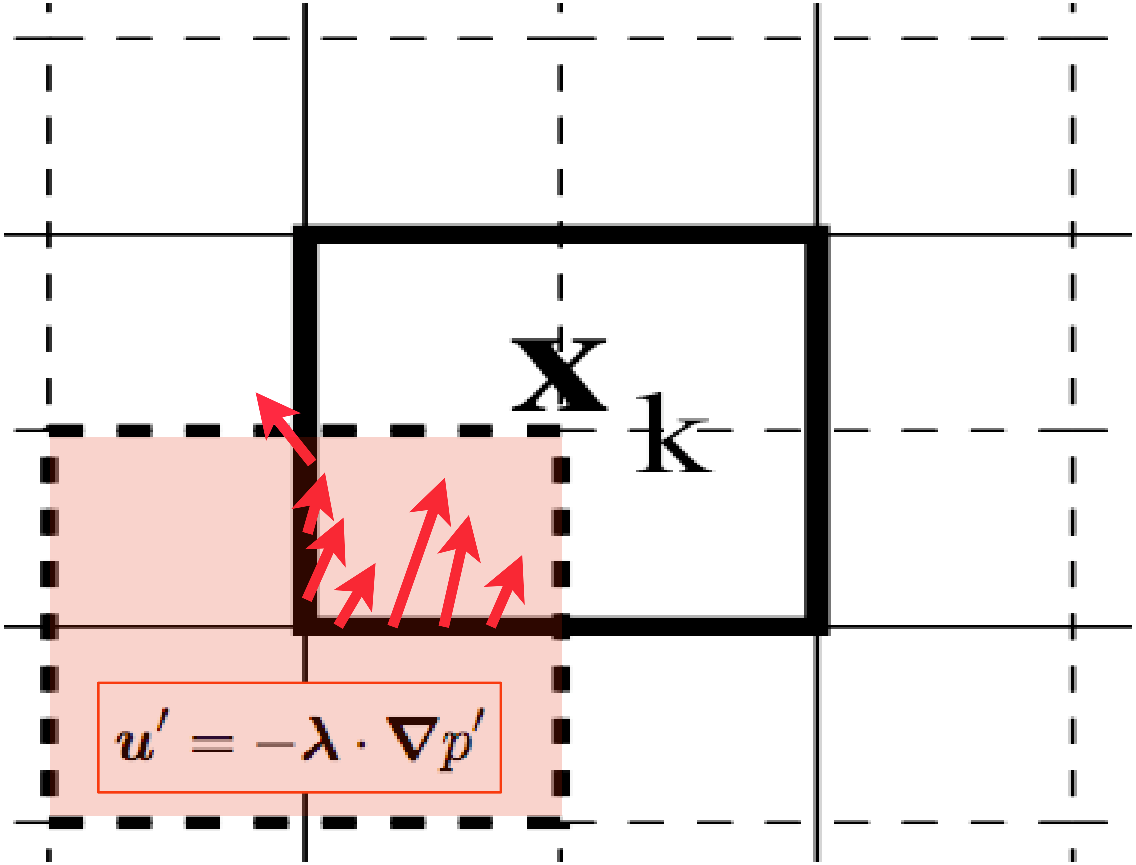




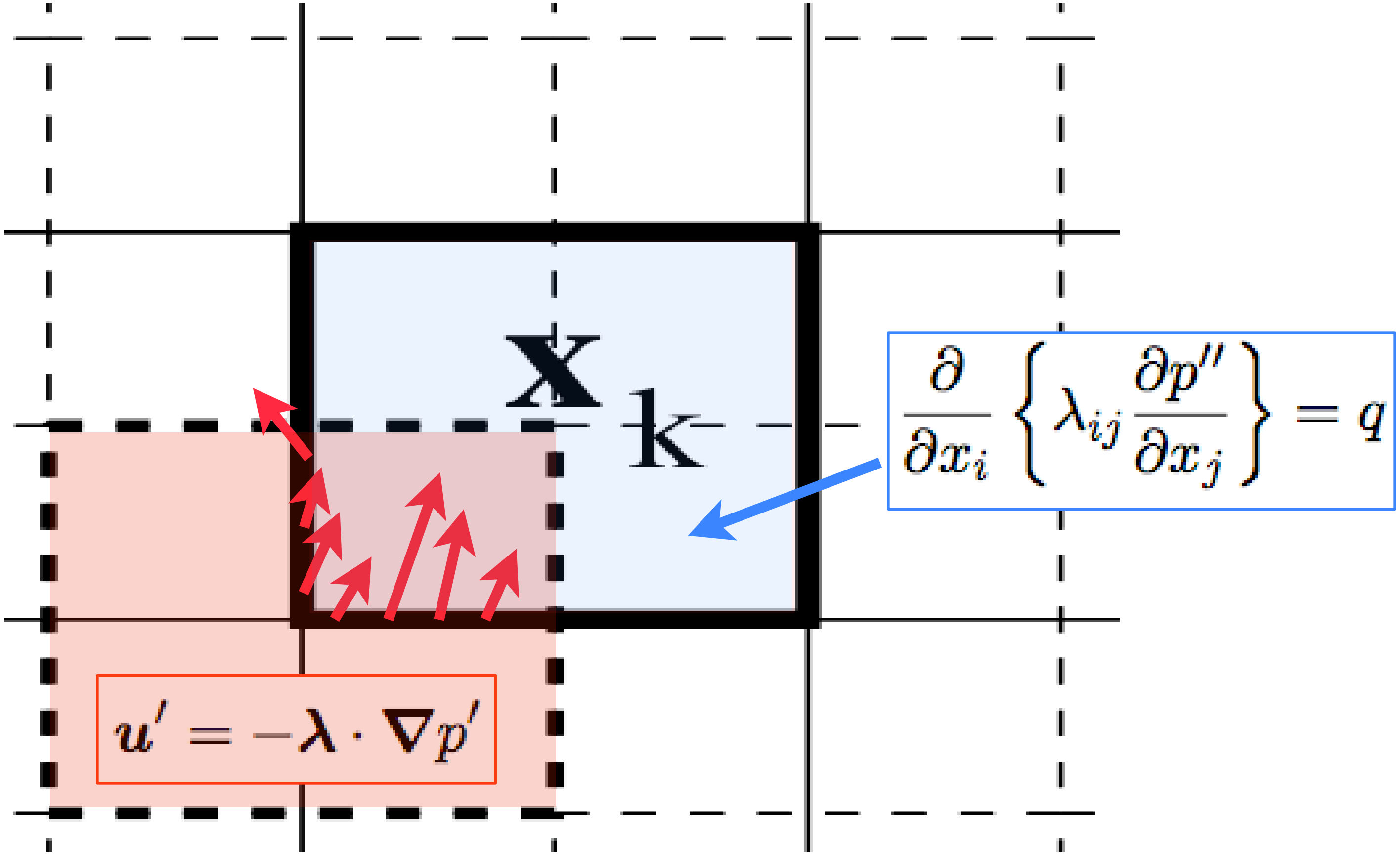
Local Conservativity



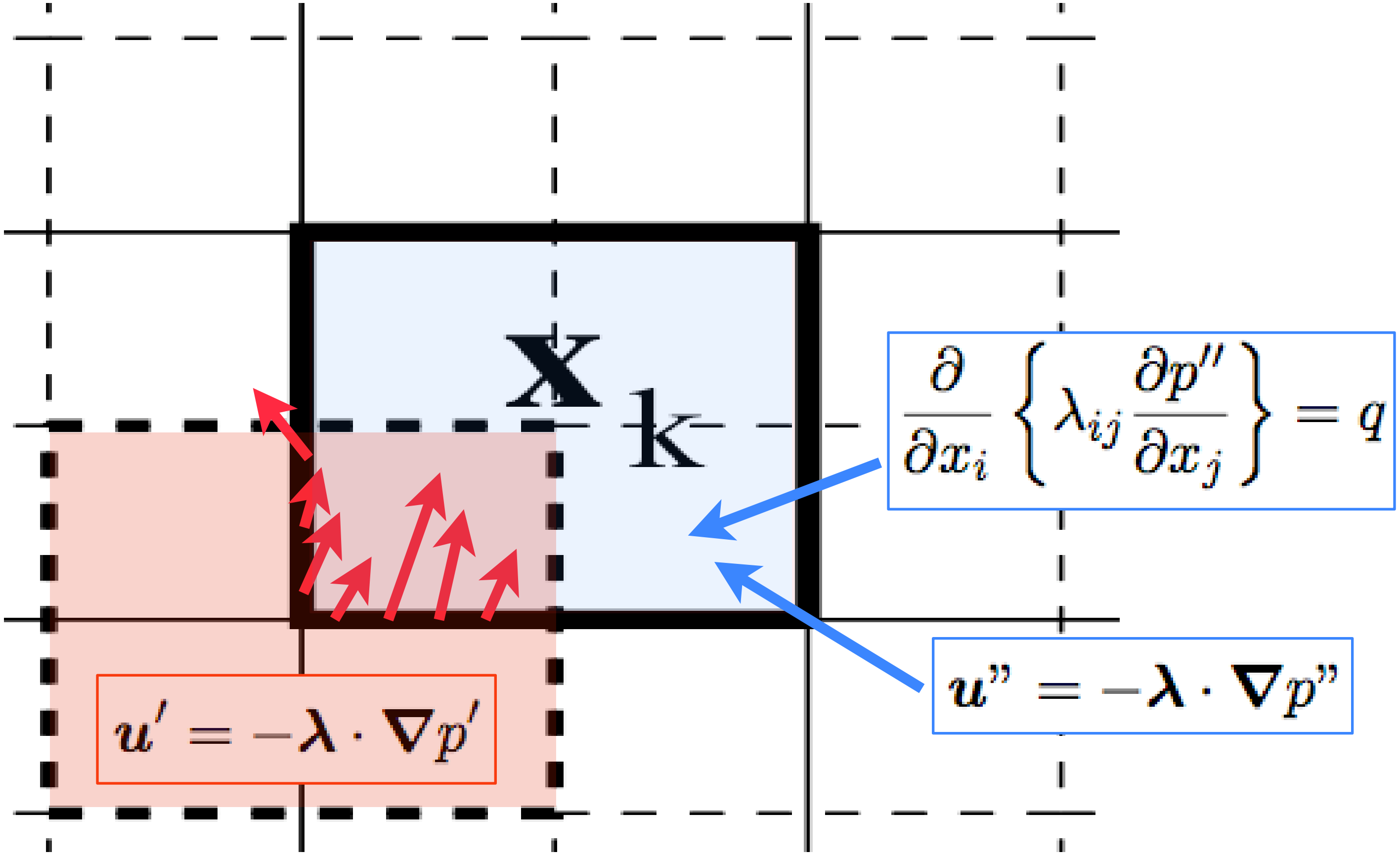
Local Conservativity



Local Conservativity

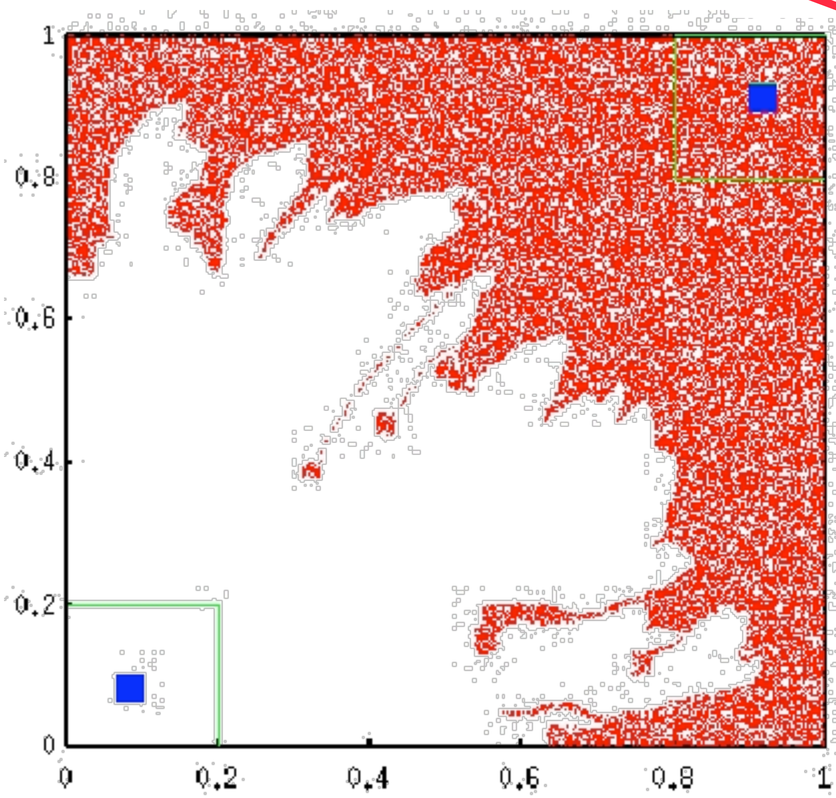


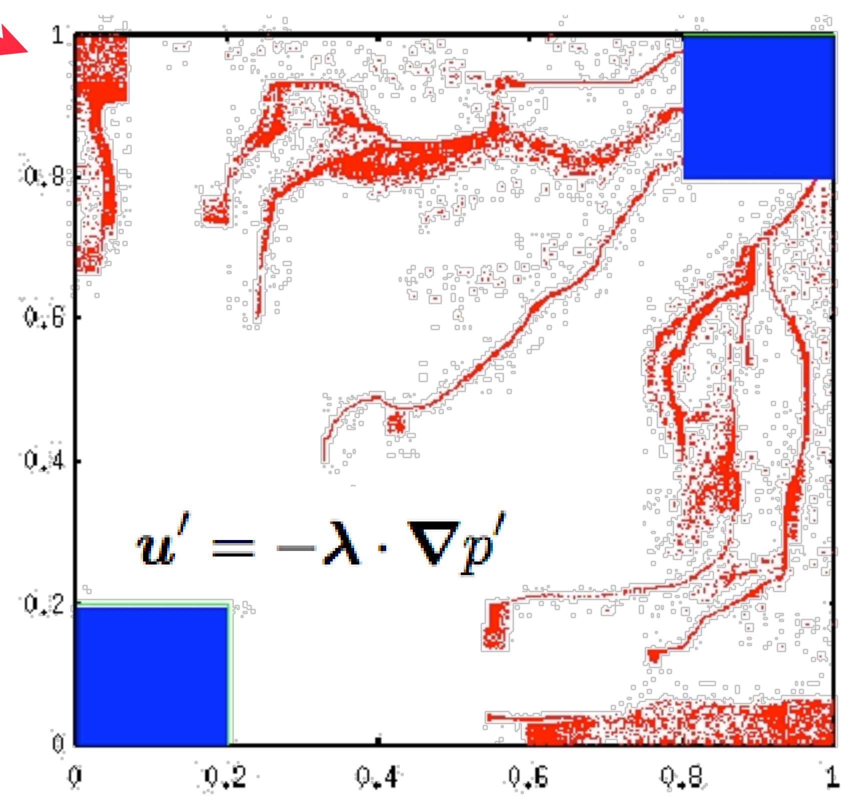
Local Conservativity



Local Conservativity

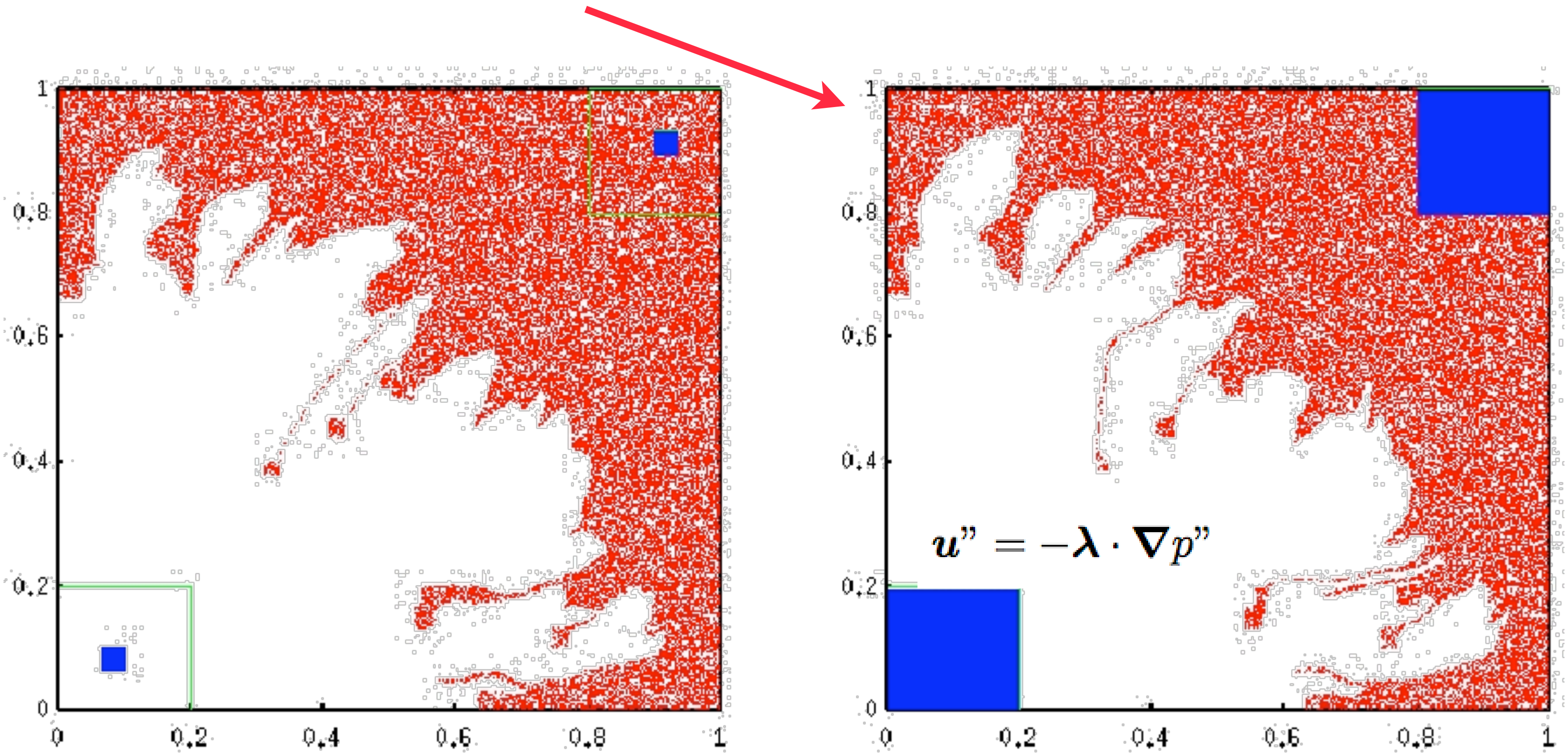
with the approximation $p(\boldsymbol{x}) \approx p'(\boldsymbol{x}) = \sum_{h=1}^{N} \left\{ \sum_{k=1}^{M} \left[\Phi_k^h \bar{p}_k \right] + \Phi^h \right\}$



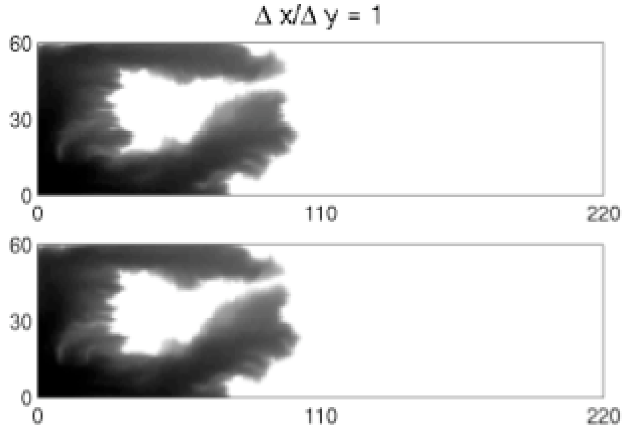


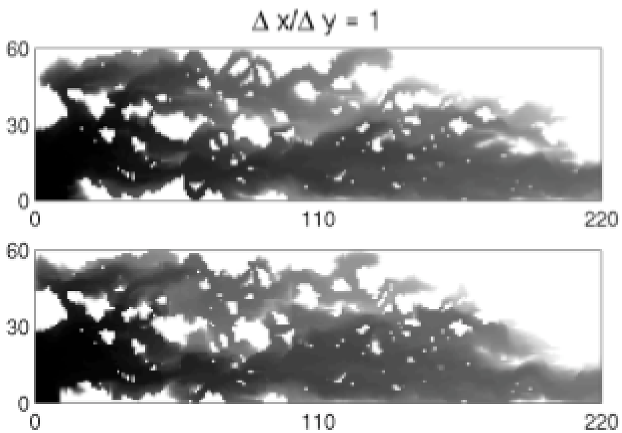
Local Conservativity

with the conservative velocity reconstruction

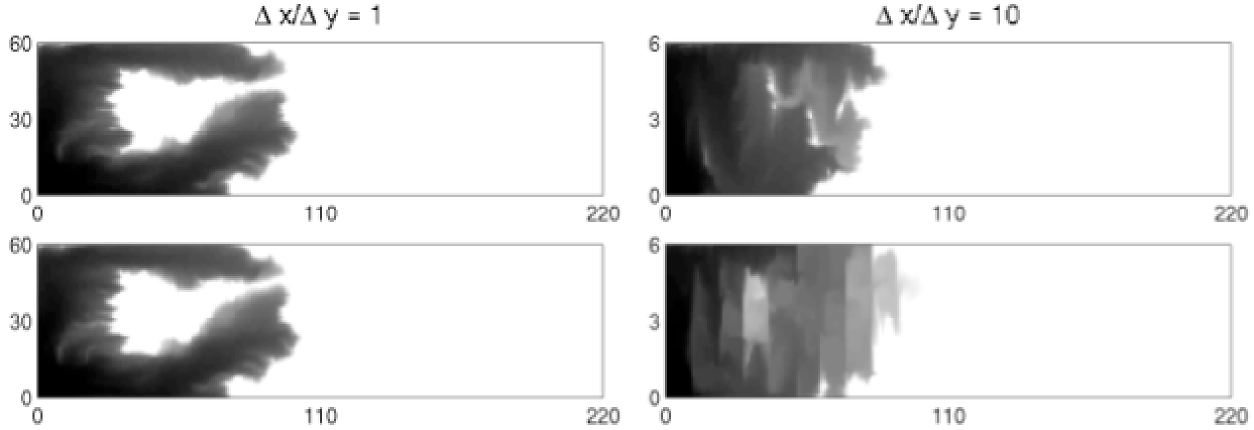


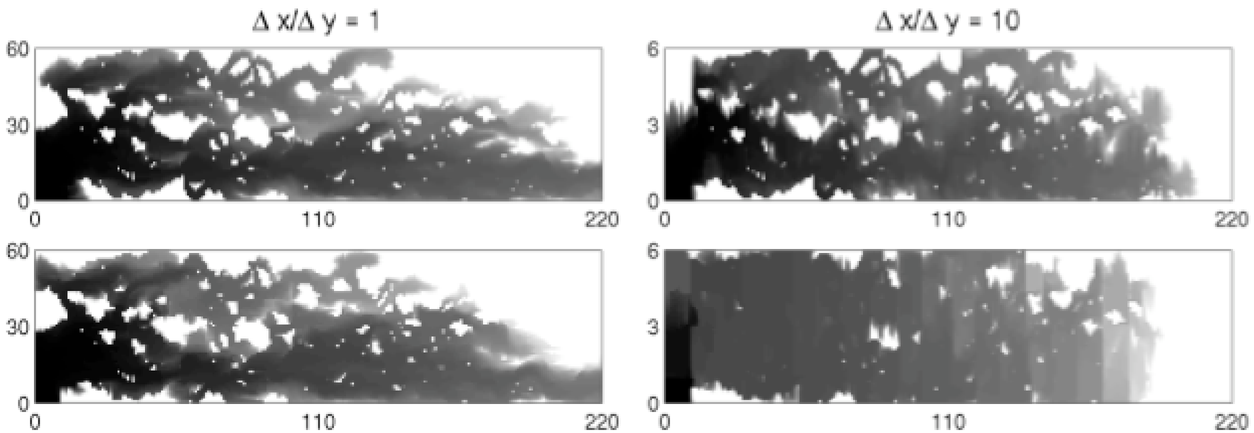
Example: Comparative SPE 10 Test Case



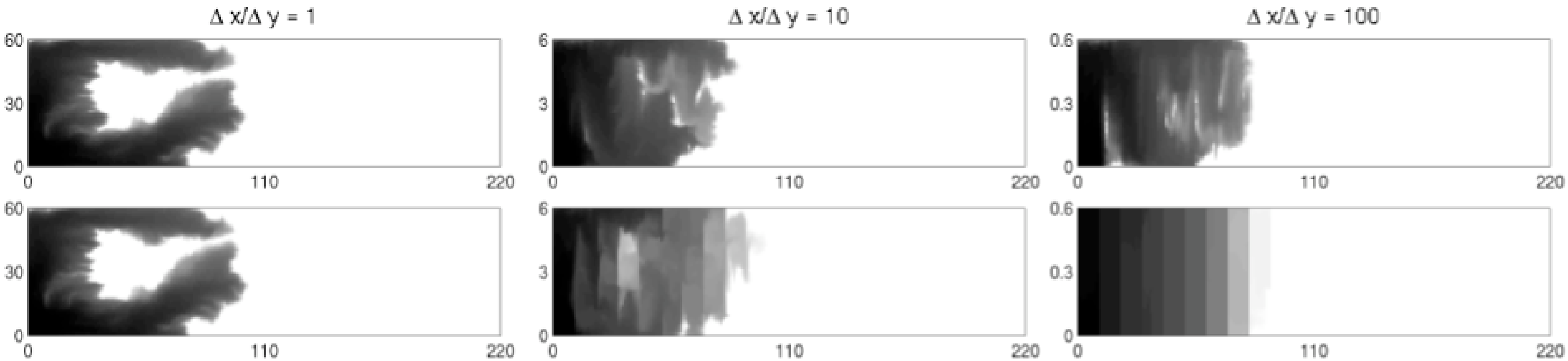


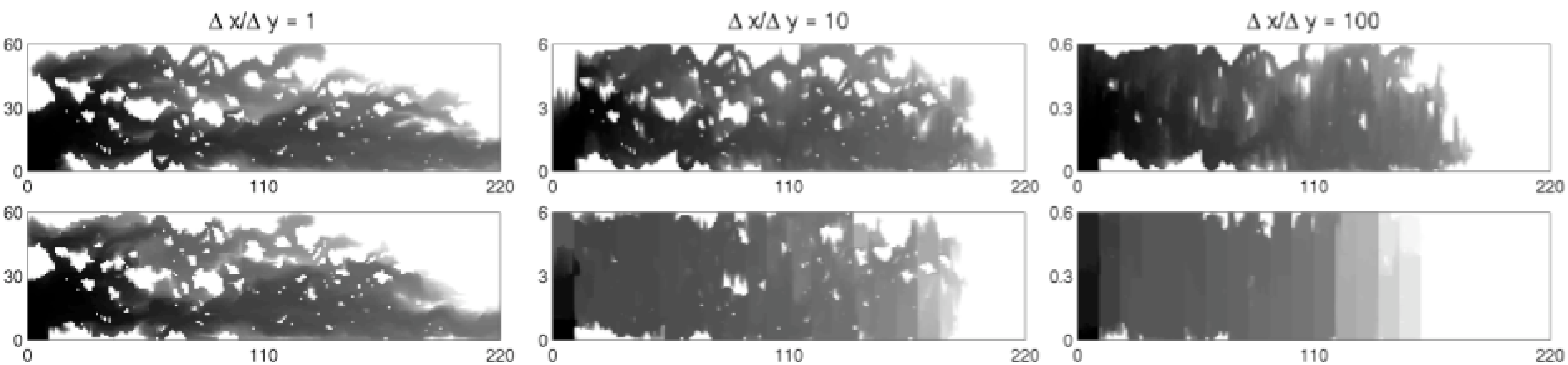
Problems with the MSFV Method: Anisotropy



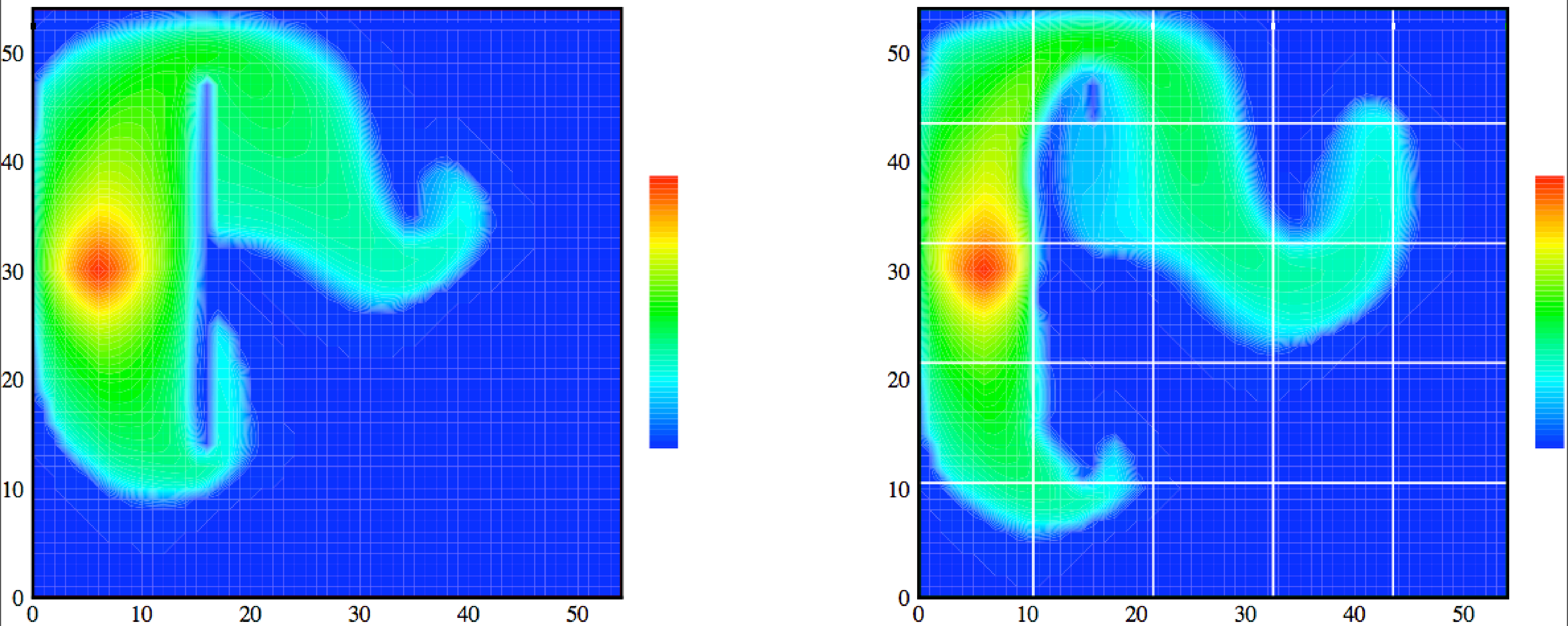


Problems with the MSFV Method: Anisotropy





Problems with the MSFV Method: Shale Layers



Summary I

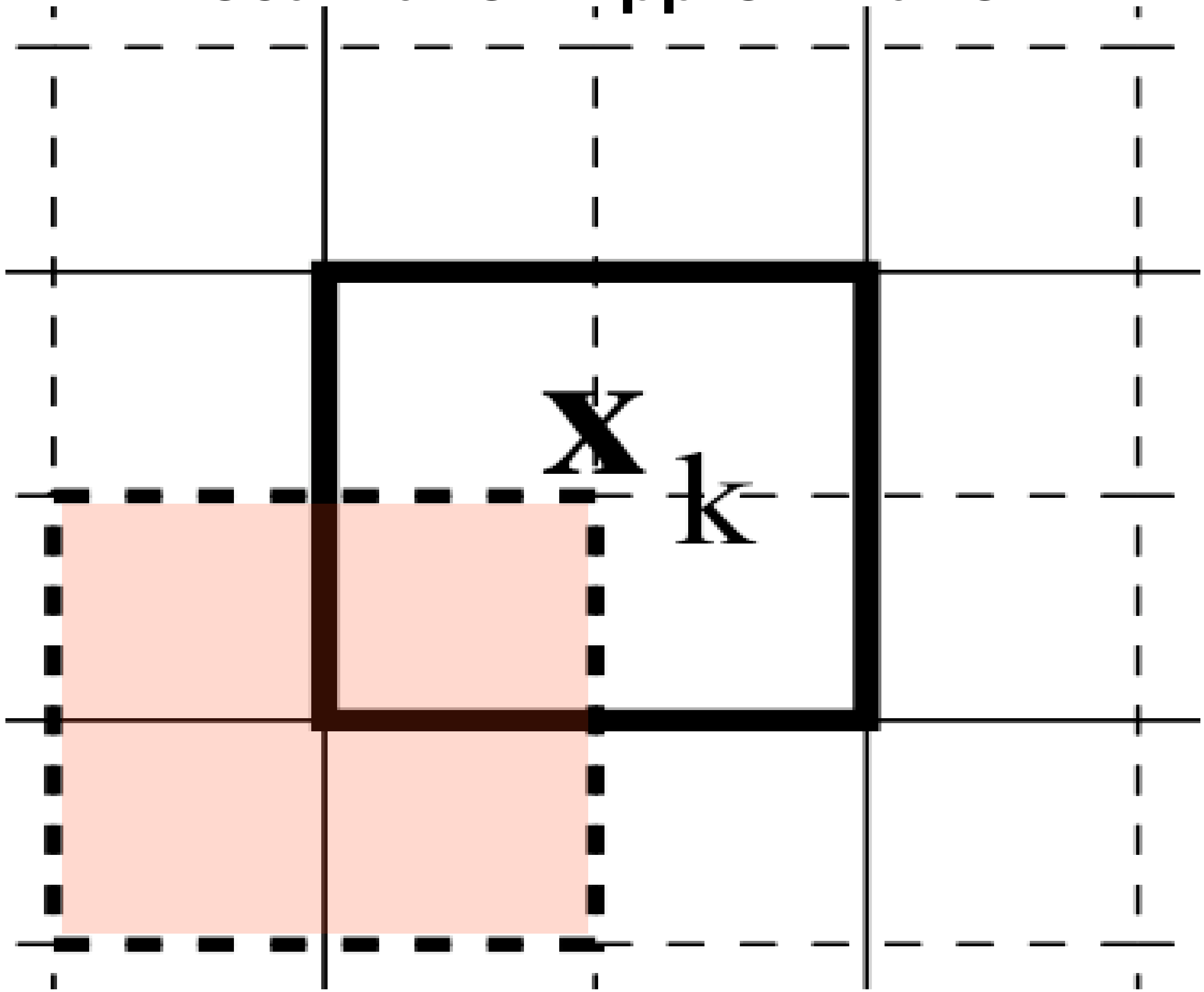
- Adaptive parametrized solution => few dofs => efficient
- Consistent weak formulation
- Conservative (important for transport)
- Only approximation at local boundaries
- Good approximation for elliptic or parabolic multiscale problems
- Can be combined with gravity, wells, fractures, ...
- Ideal for multi-physics and multi-numerics problems, e.g.

  NS/Darcy, wells, fractures, adaptive transport, ...

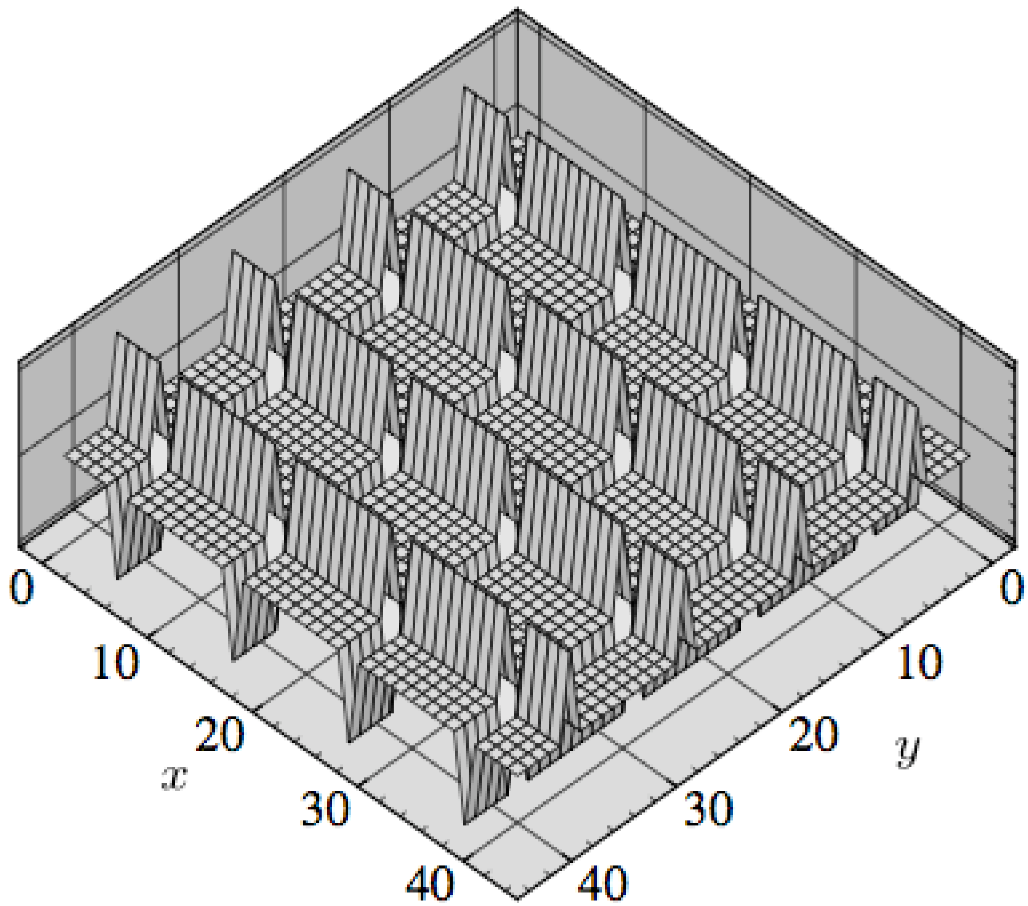
- Problems with strong anisotropy and shale layers

  => iterative MSFV method

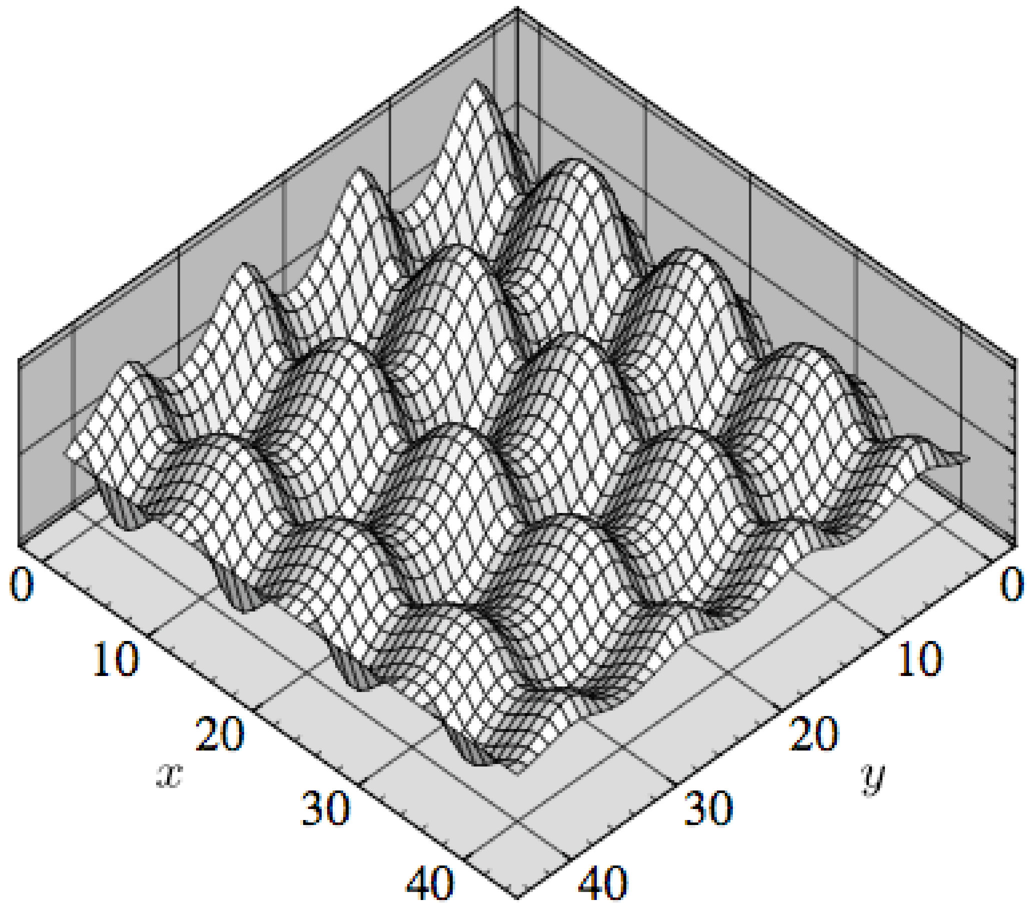
Localization Approximation



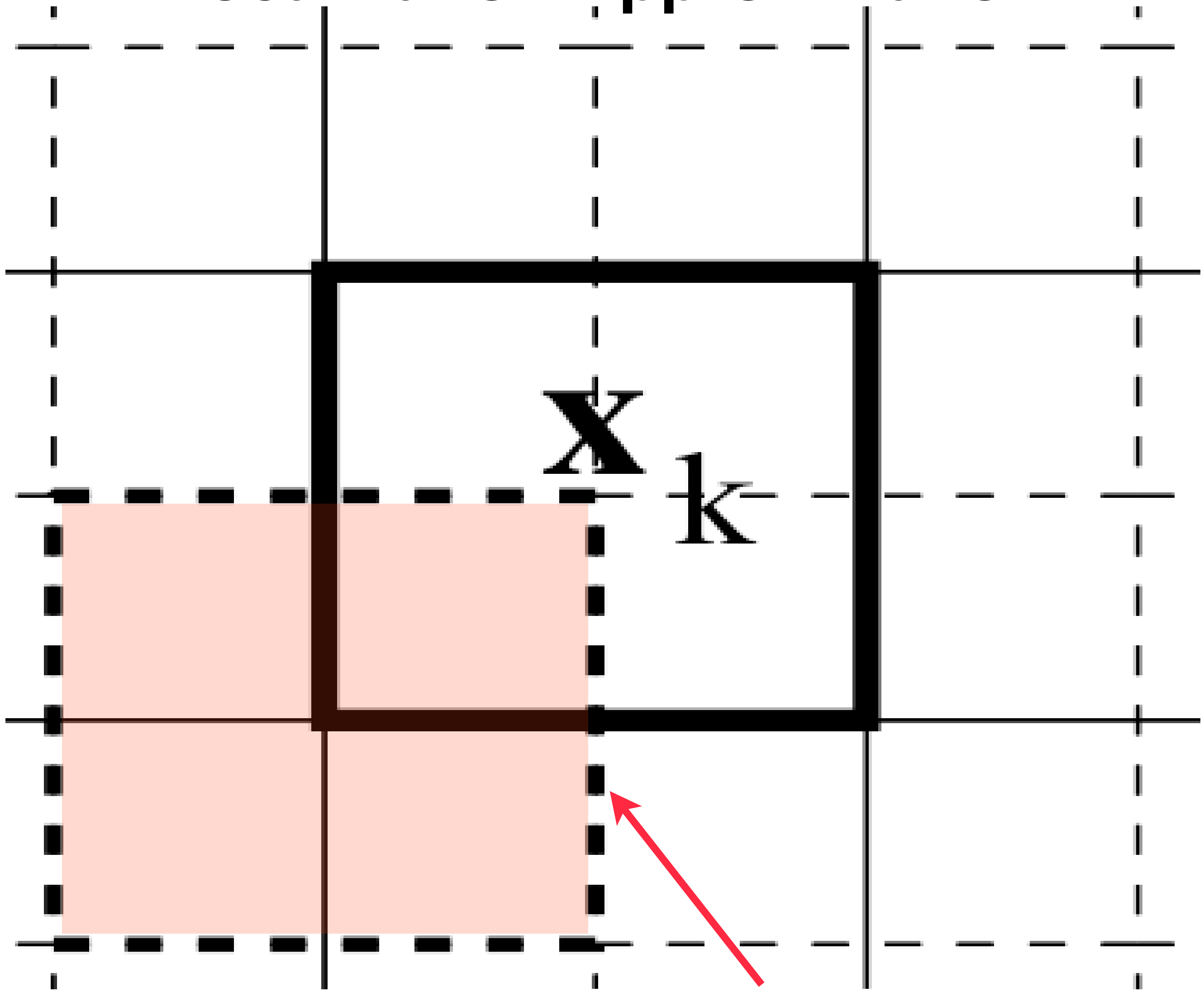
Residuum



Smoothed Residuum

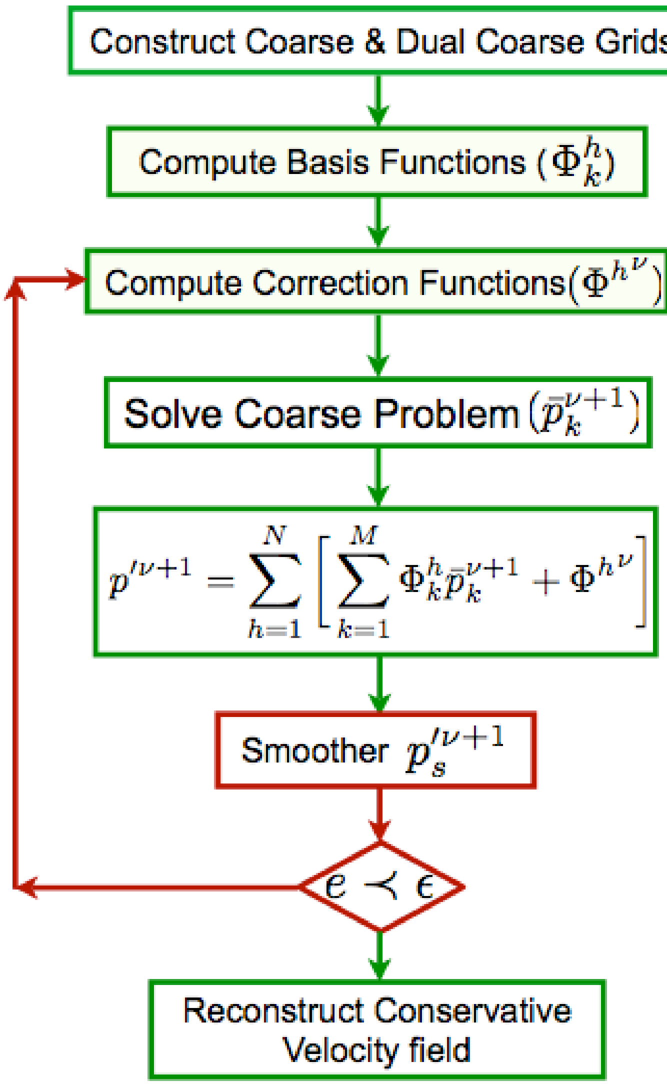


Localization Approximation

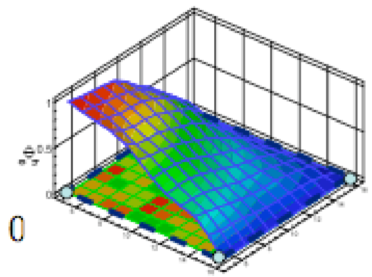


$$r^{h^{(t)}} = (\tilde{\boldsymbol{n}}^h \cdot \nabla)((\boldsymbol{\lambda} \cdot \nabla p_s'^{(t)}) \cdot \tilde{\boldsymbol{n}}^h)$$

Iterative Multiscale Method (iMSFV) (Hajibeygi et al.)



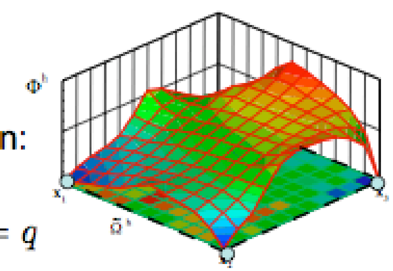
Basis Functions:



$$-\nabla \cdot (\underline{\underline{\lambda}}_t \cdot \nabla \Phi_k^h) = 0$$

$$BC : (\tilde{n}^h \cdot \nabla)((\underline{\underline{\lambda}}_t \cdot \nabla \Phi_k^h) \cdot \tilde{n}^h) = 0$$

Correction Function:

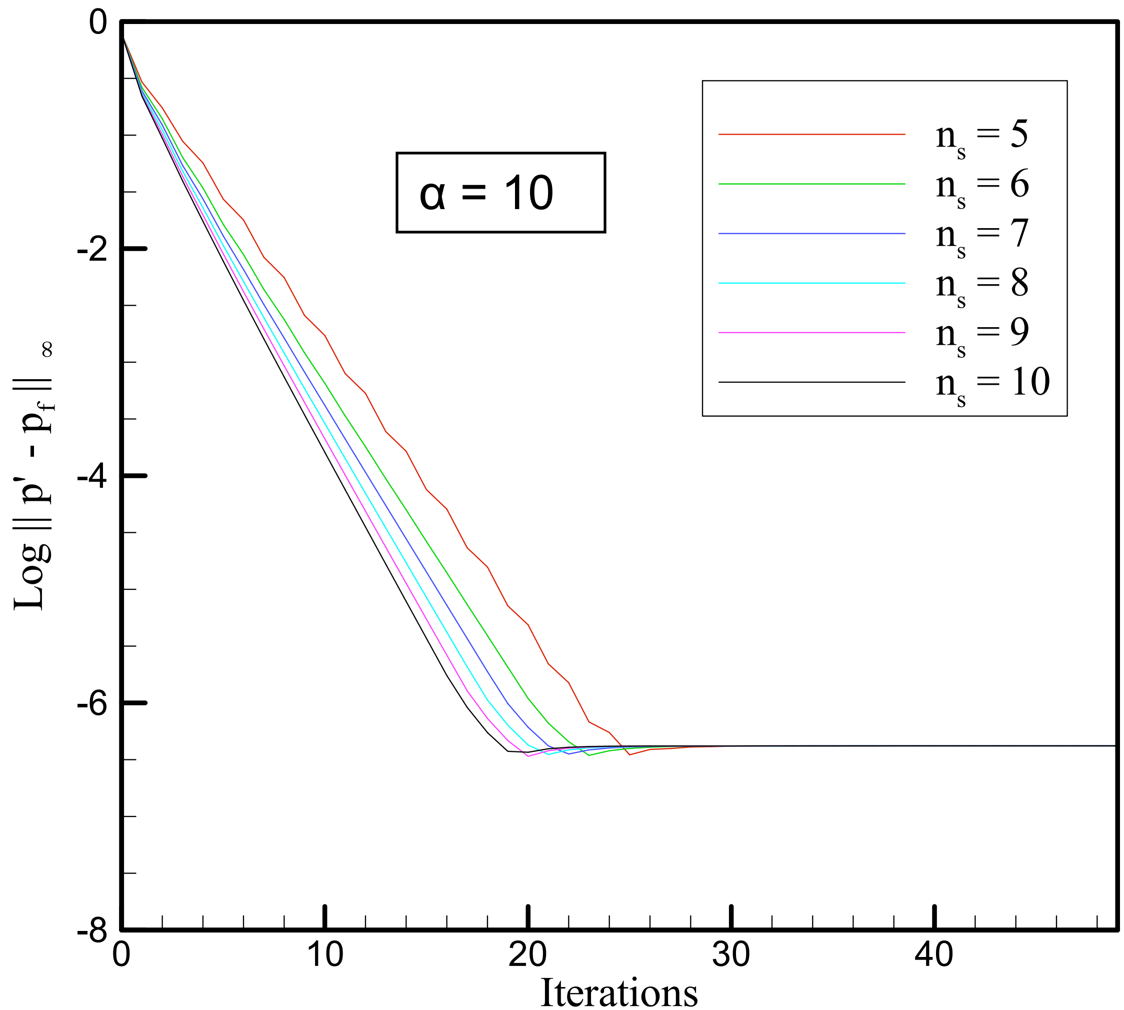


$$-\nabla \cdot (\underline{\underline{\lambda}}_t \cdot \nabla \Phi^{h^\nu}) = q$$

$$BC : (\tilde{n}^h \cdot \nabla)((\underline{\underline{\lambda}}_t \cdot \nabla \Phi^{h^\nu}) \cdot \tilde{n}^h) = (\nabla \cdot \tilde{n}^h)((\underline{\underline{\lambda}}_t \cdot \nabla p_s'^\nu) \cdot \tilde{n}^h)$$

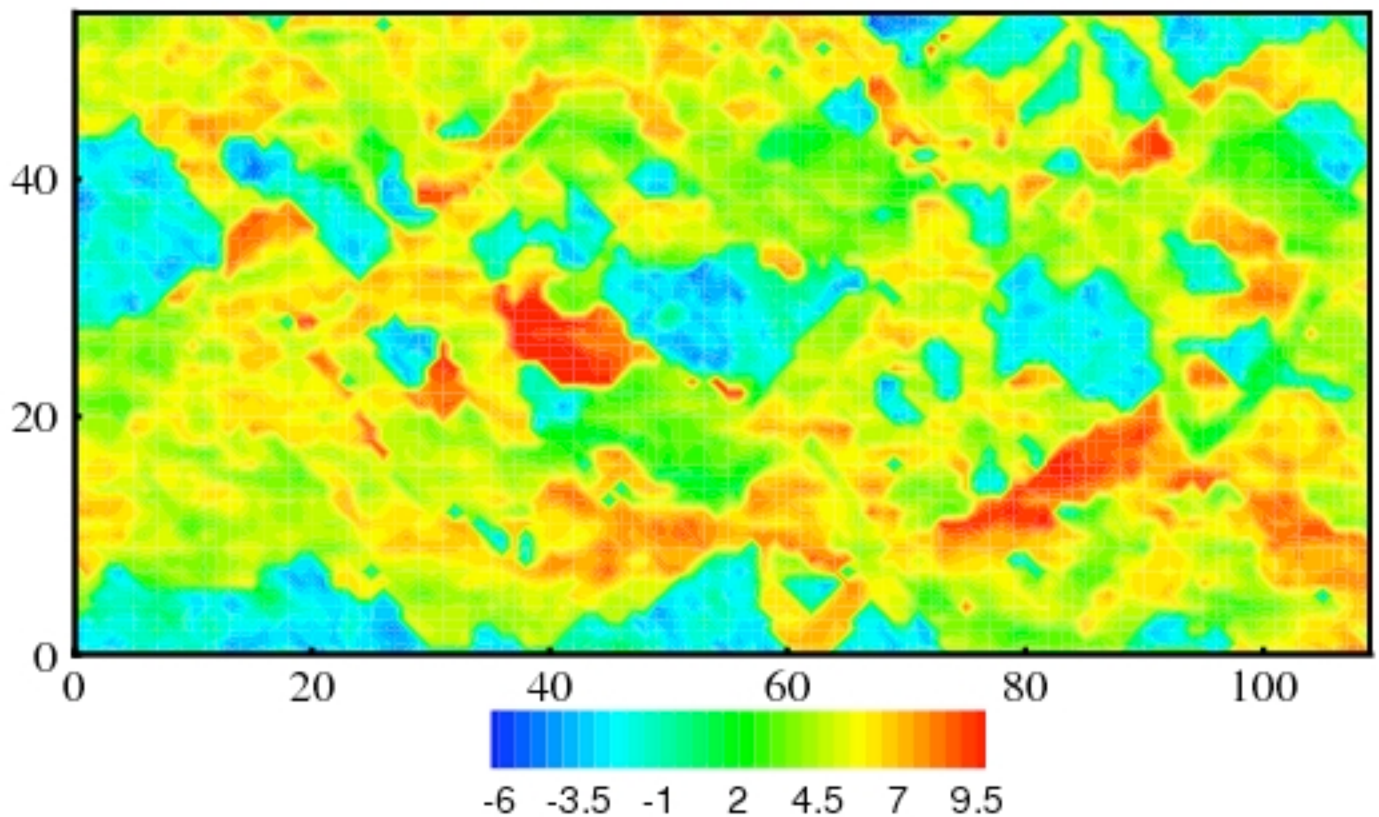
Iterative Multiscale Method (iMSFV) (Hajibeygi et al.)

Shale Layers with Anisotropic Permeability Field



Iterative Multiscale Method (iMSFV) (Hajibeygi et al.)

SPE 10 Bottom Layer Test Case

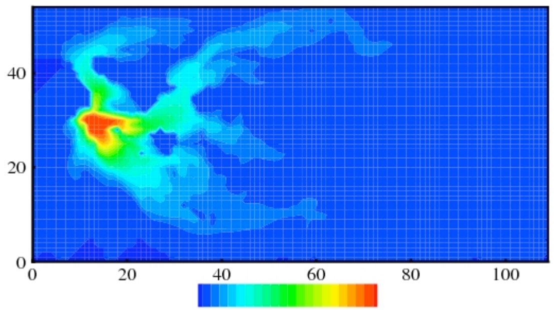


SPE 10 Bottom Layer Test Case

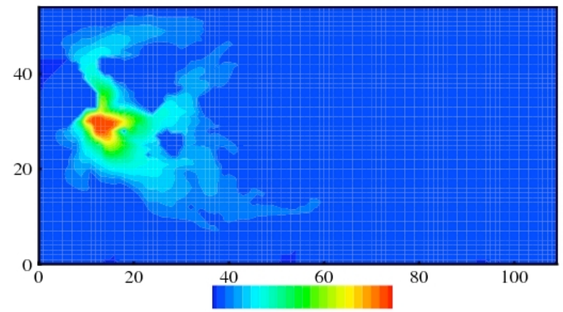
α = 1

Viscosity ratio = 10

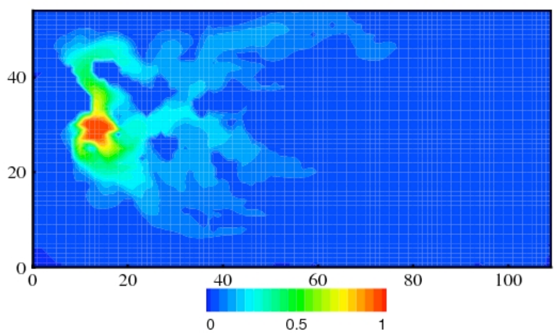
Saturation Field



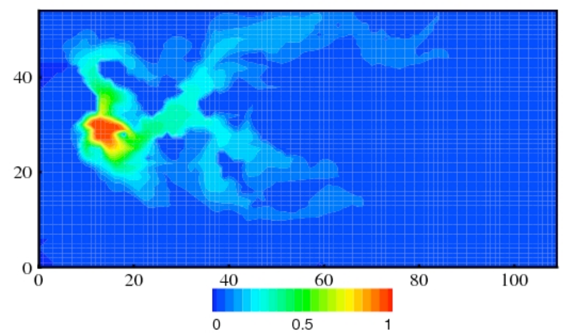
Fine Scale



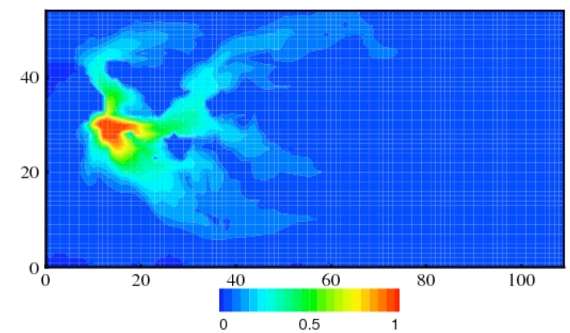
MSFV



iMSFV-a



iMSFV-b



iMSFV-c

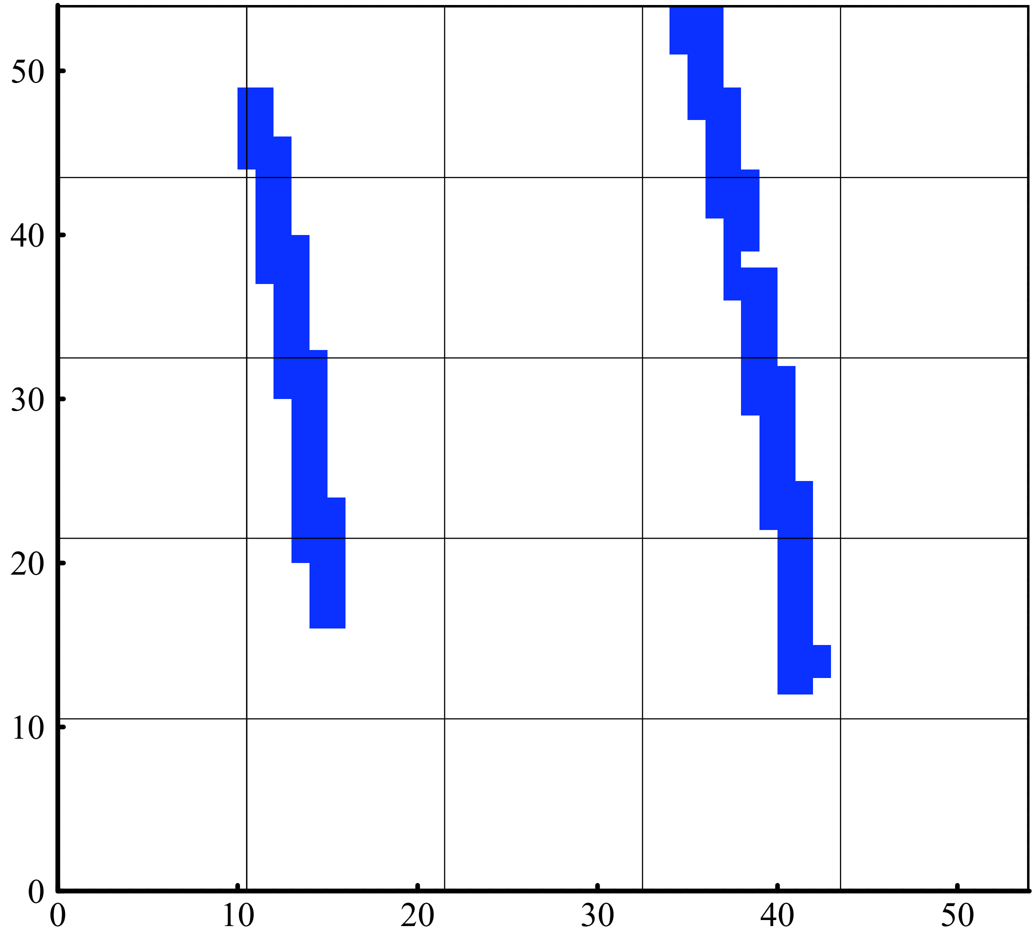
(a) Using the first time step smoothed pressure field in all next time steps.

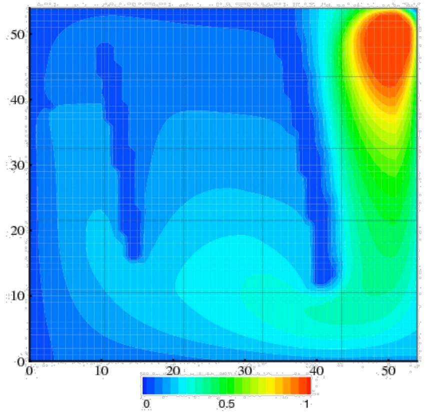
(b) Updating smoothed pressure field every 10 time steps by applying 1 LR.

(c) Updating smoothed pressure field every 10 time steps by applying 10 LR.

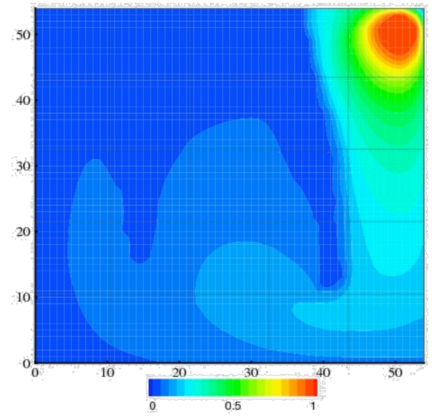
Shale Layer Test Case

$K_{shale} = 10^{-8}$ & $K_m = 100$





Fine Scale

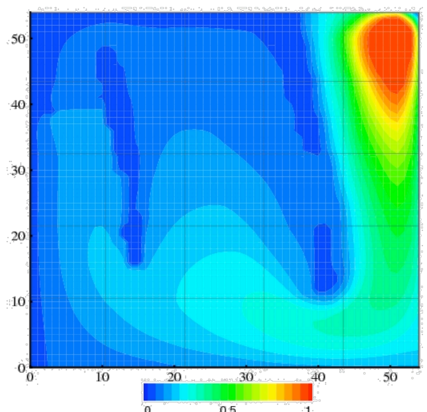


MSFV

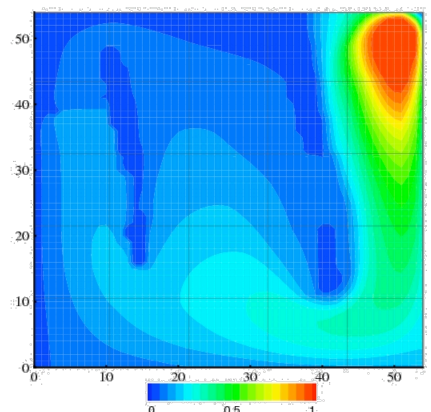
α = 1

Viscosity ratio = 10

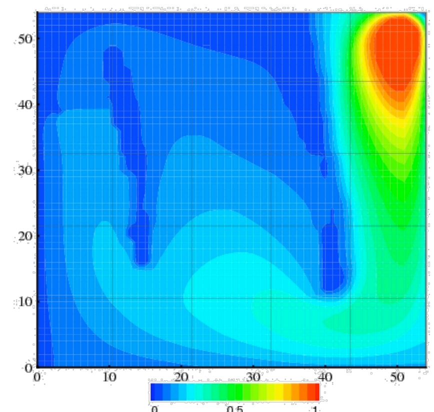
Saturation Field



iMSFV-a



iMSFV-b

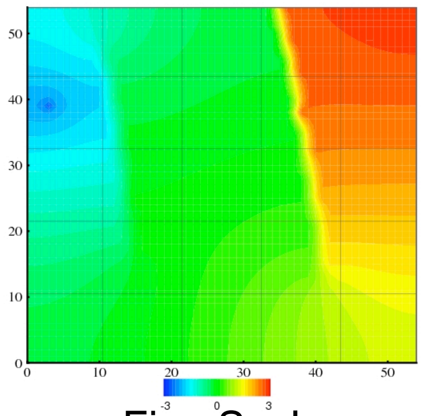


iMSFV-c

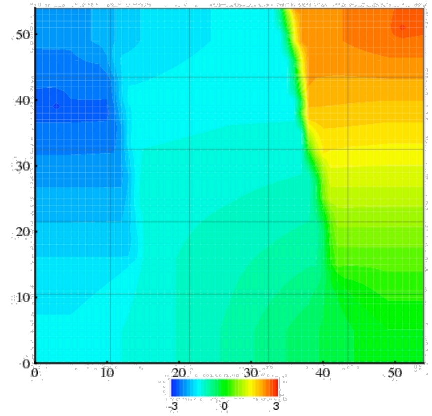
(a) Using the first time step smoothed pressure field in all next time steps.

(b) Updating smoothed pressure field every 100 time steps by applying 1 LR.

(c) Updating smoothed pressure field every 10 time steps by applying 1 LR.



Fine Scale

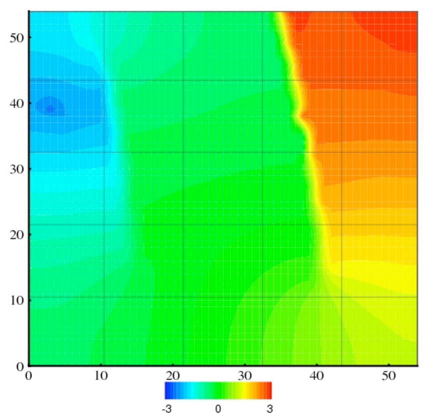


MSFV

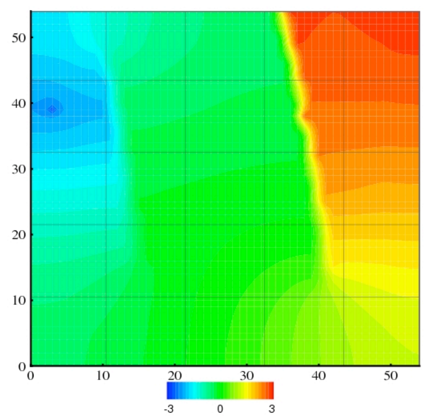
α = 1

Viscosity ratio = 10

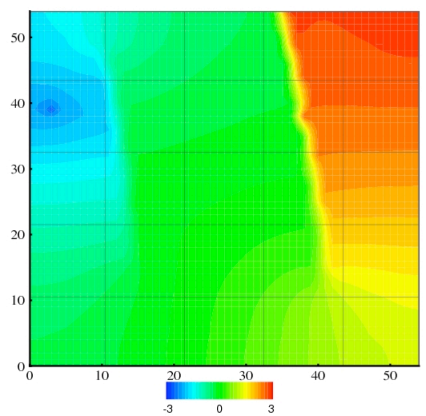
Pressure Field



iMSFV-a



iMSFV-b

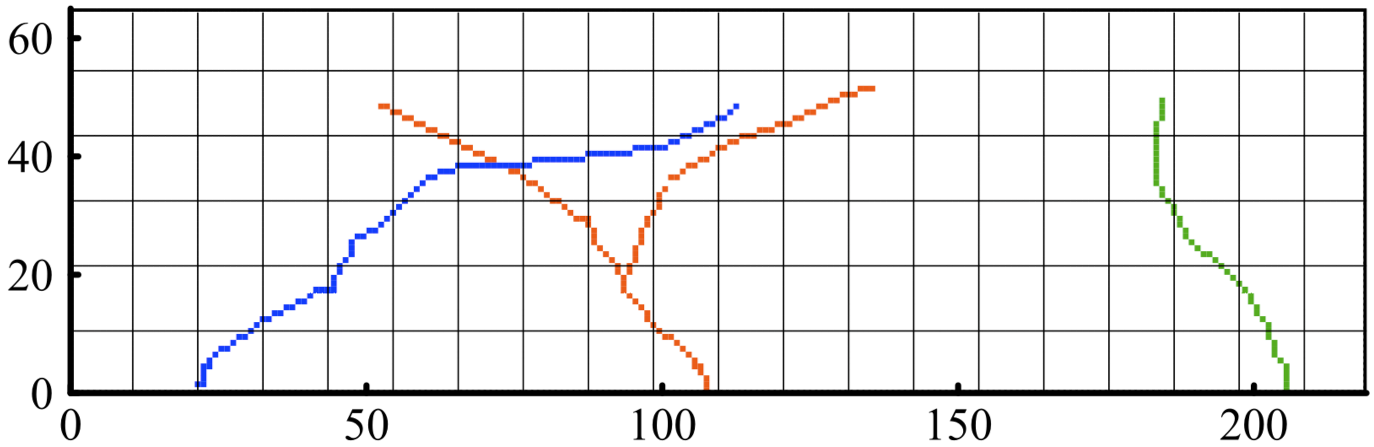


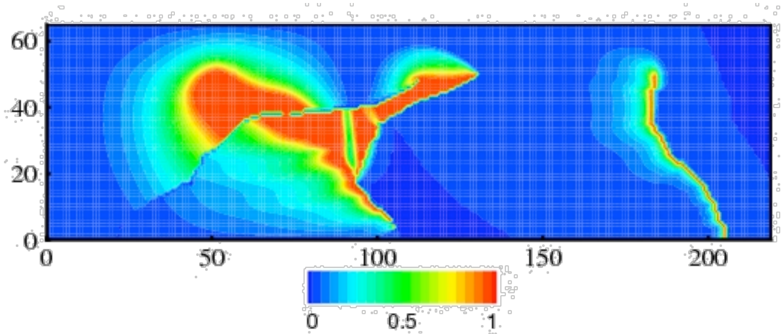
iMSFV-c

(a) Using the first time step smoothed pressure field in all next time steps.

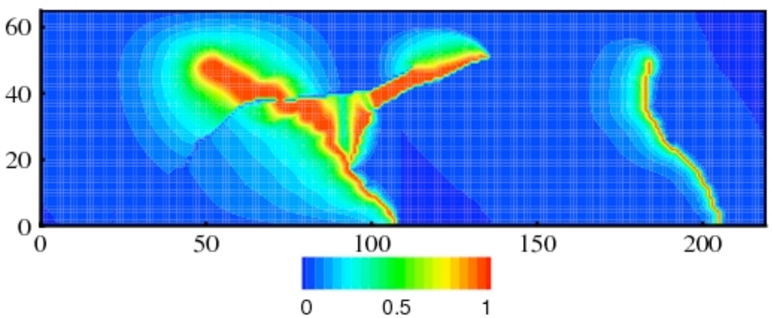
(b) Updating smoothed pressure field every 100 time steps by applying 1 LR.

(c) Updating smoothed pressure field every 10 time steps by applying 1 LR.

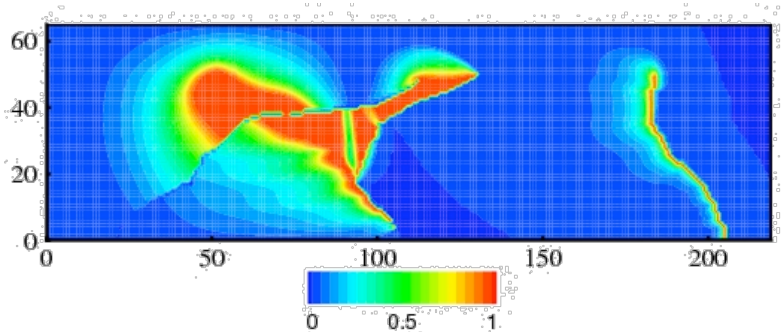

60
40
20
0
0
50
100
150
200



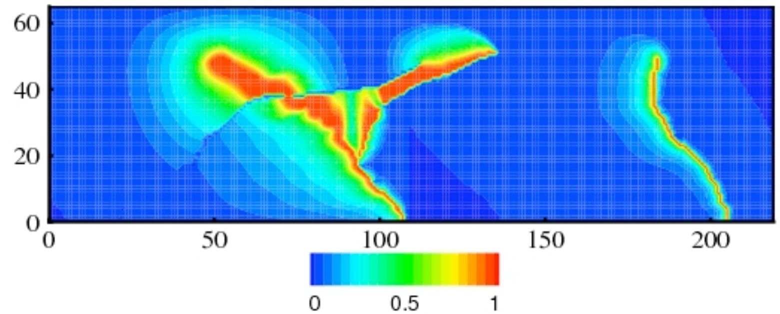
Fine Scale



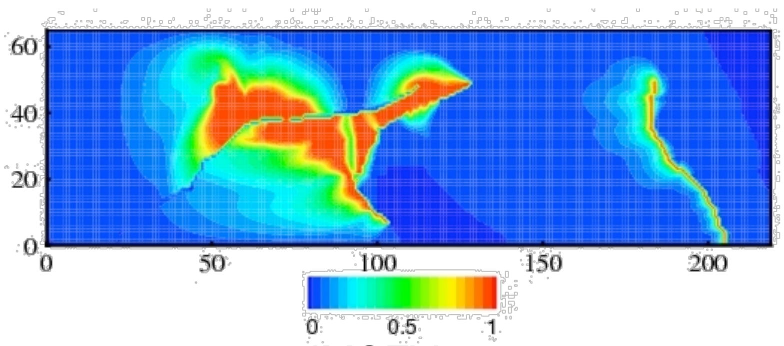
MSFV



Fine Scale

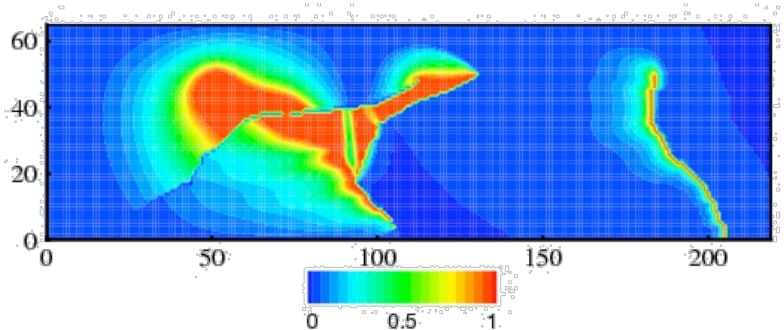


MSFV

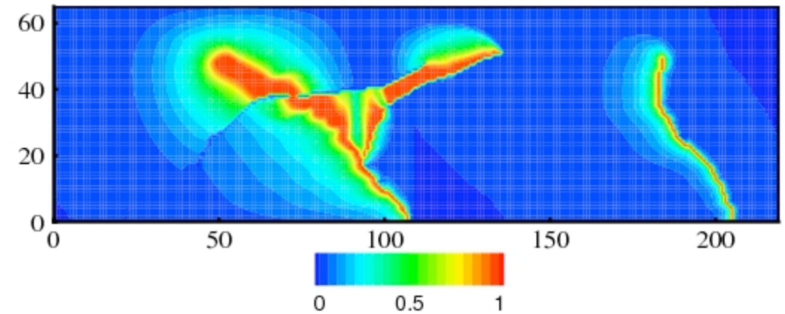


iMSFV-a

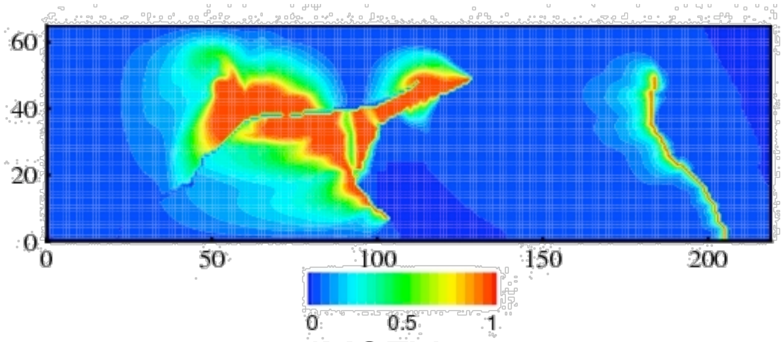
(a) Using the first time step smoothed pressure field in all next time steps.



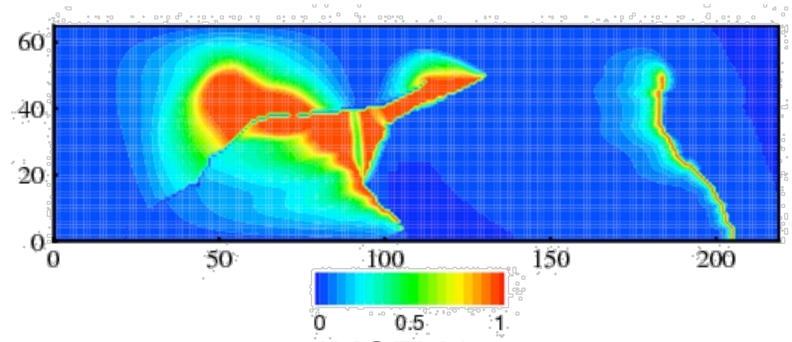
Fine Scale



MSFV



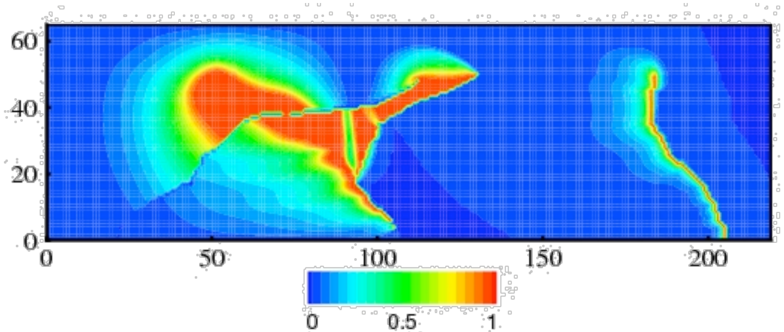
iMSFV-a



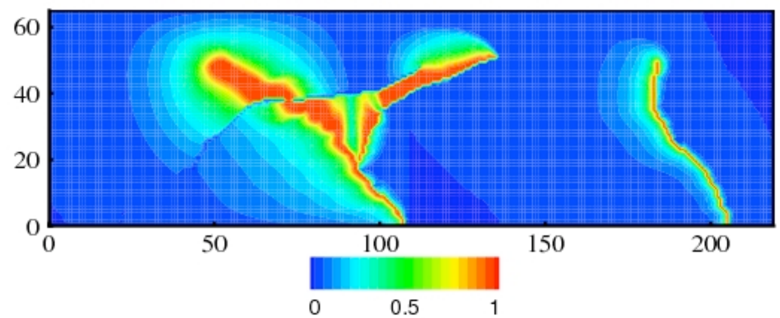
iMSFV-b

(a) Using the first time step smoothed pressure field in all next time steps.

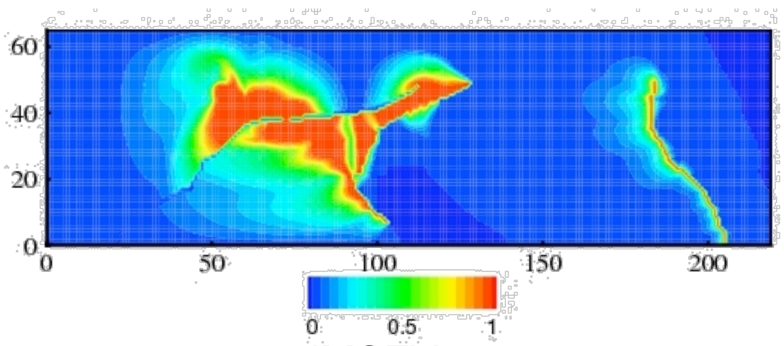
(b) Updating smoothed pressure field every 100 time steps by applying 1 LR.



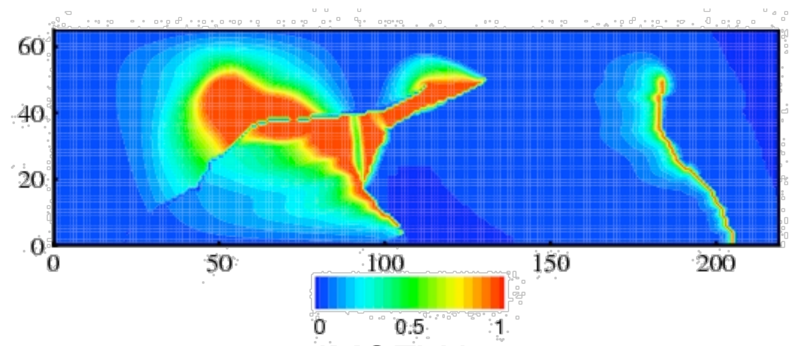
Fine Scale



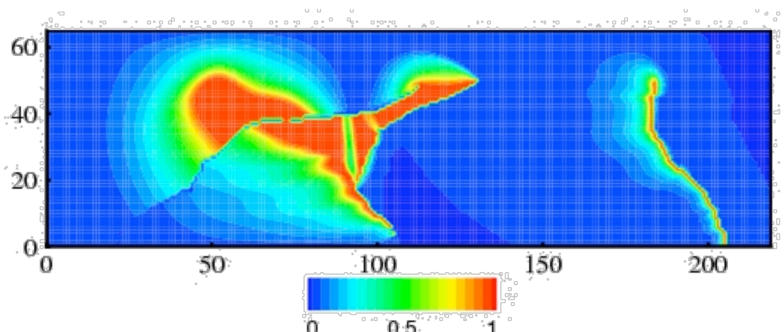
MSFV



iMSFV-a



iMSFV-b



iMSFV-c

(a) Using the first time step smoothed pressure field in all next time steps.

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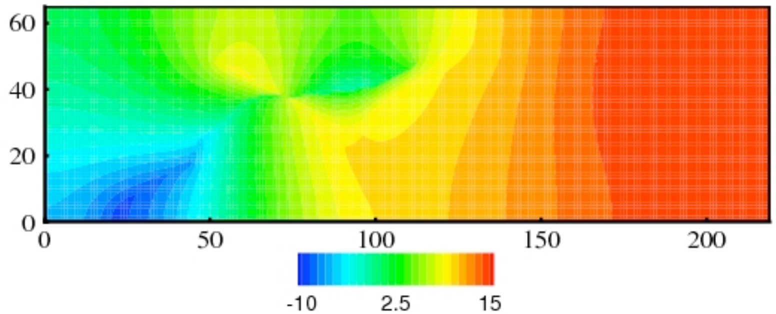
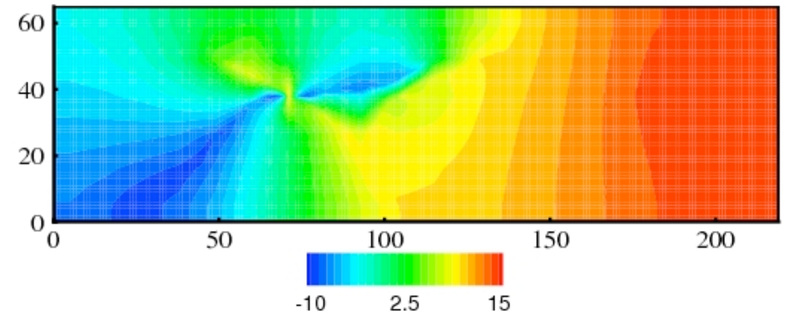
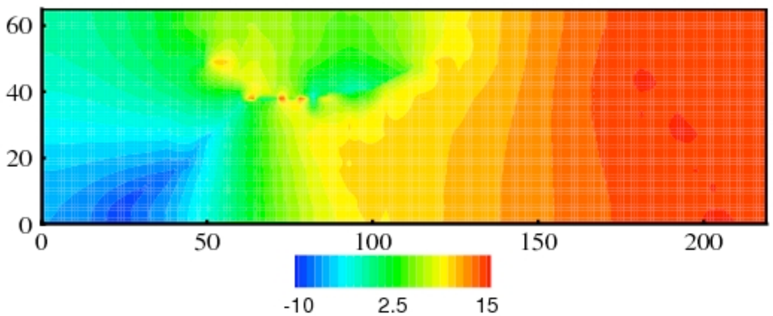
(c) Updating smoothed pressure field every 10 time steps by applying 1 LR.



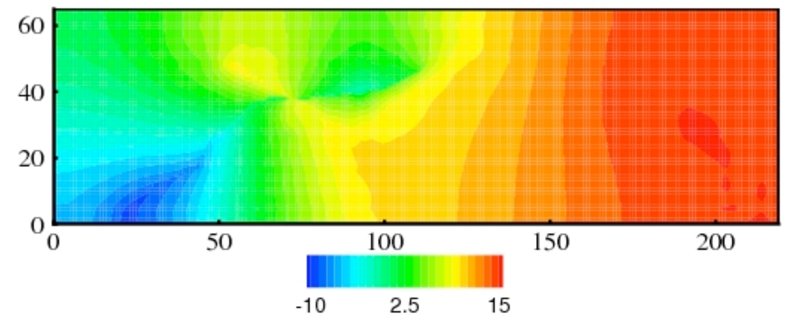
Fine Scale



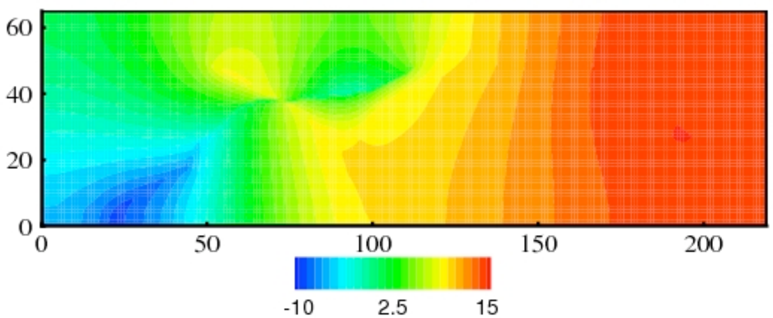
MSFV



iMSFV-a



iMSFV-b



iMSFV-c

(a) Using the first time step smoothed pressure field in all next time steps.

(b) Updating smoothed pressure field every 100 time steps by applying 1 LR.

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Adaptivity

- Mobility

  basis- and correction functions, if:

$$\forall \boldsymbol{x} \in \tilde{\Omega} : \quad \frac{1}{1+\varepsilon_\lambda} < \frac{\lambda^{new}(\boldsymbol{x})}{\lambda^{old}(\boldsymbol{x})} < 1 + \varepsilon_\lambda$$

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- RHS

  correction function, if:

$$\forall \boldsymbol{x} \in \tilde{\Omega} : \quad \frac{1}{1+\varepsilon_q} < \frac{q^{new}(\boldsymbol{x})}{q^{old}(\boldsymbol{x})} < 1 + \varepsilon_q$$

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- Fine scale transport

  conservative reconstruction

Adaptivity

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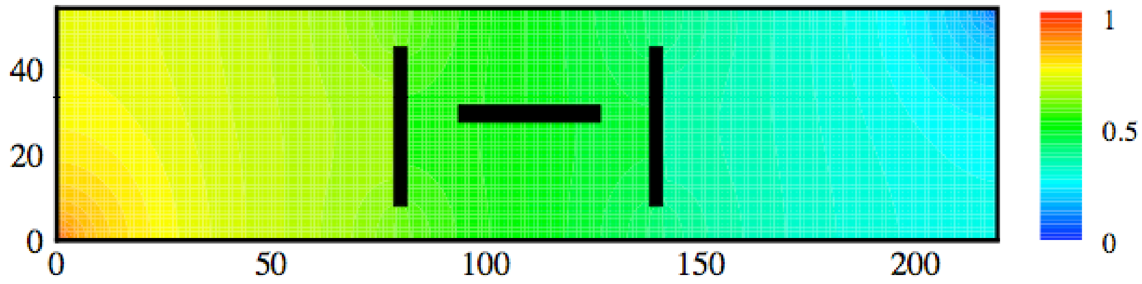
- Fine scale transport

  conservative reconstruction

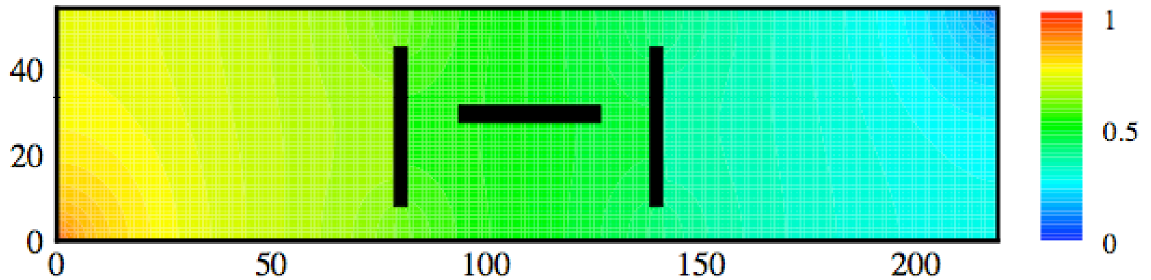
- Residuum

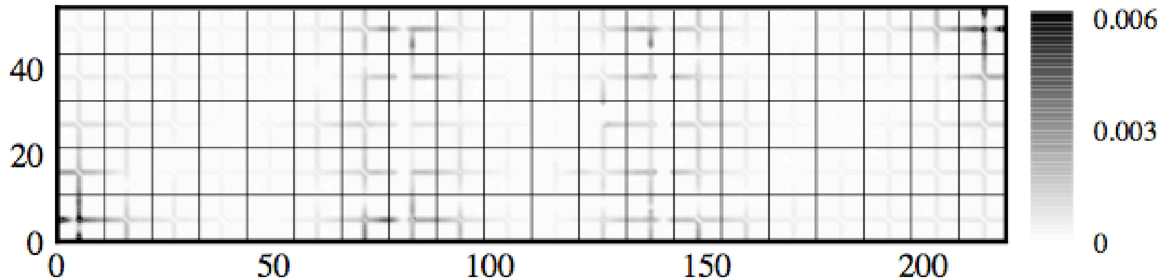
  correction function

Adaptivity: Residuum

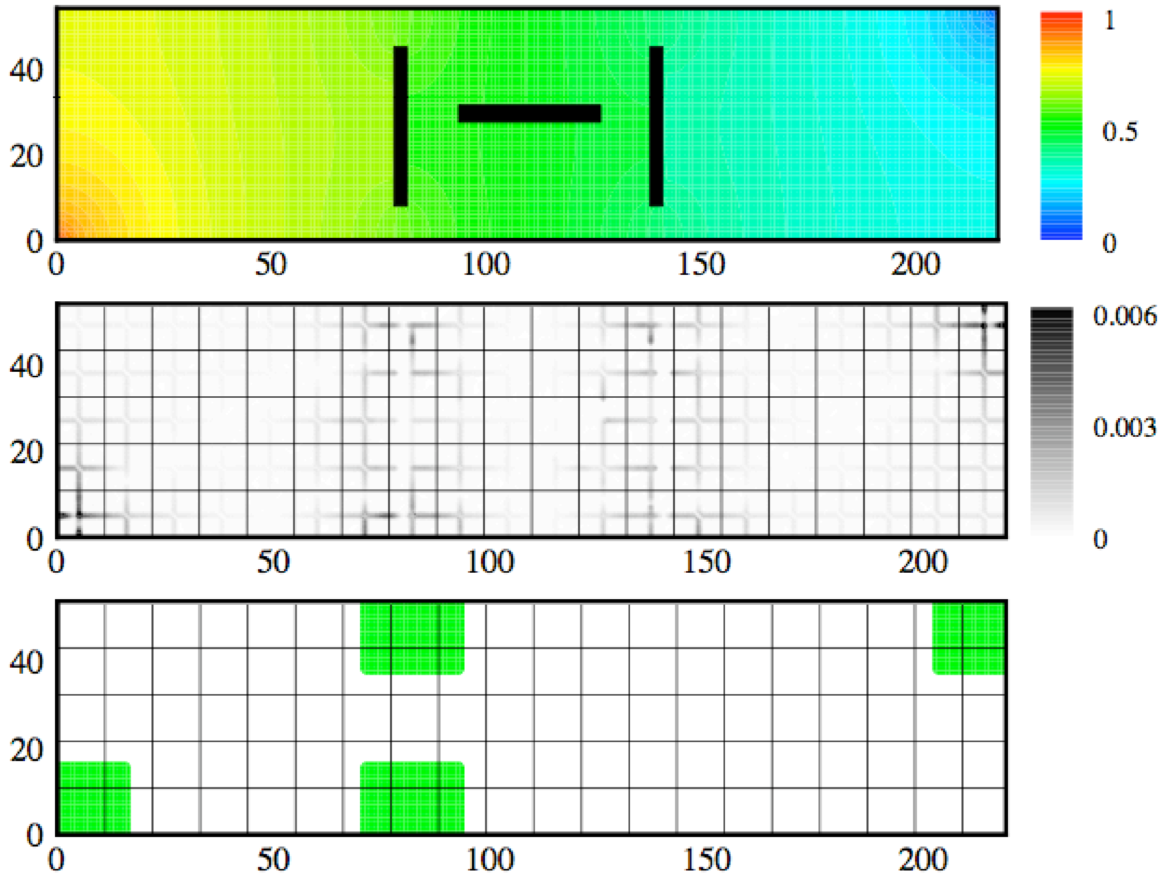


Adaptivity: Residuum

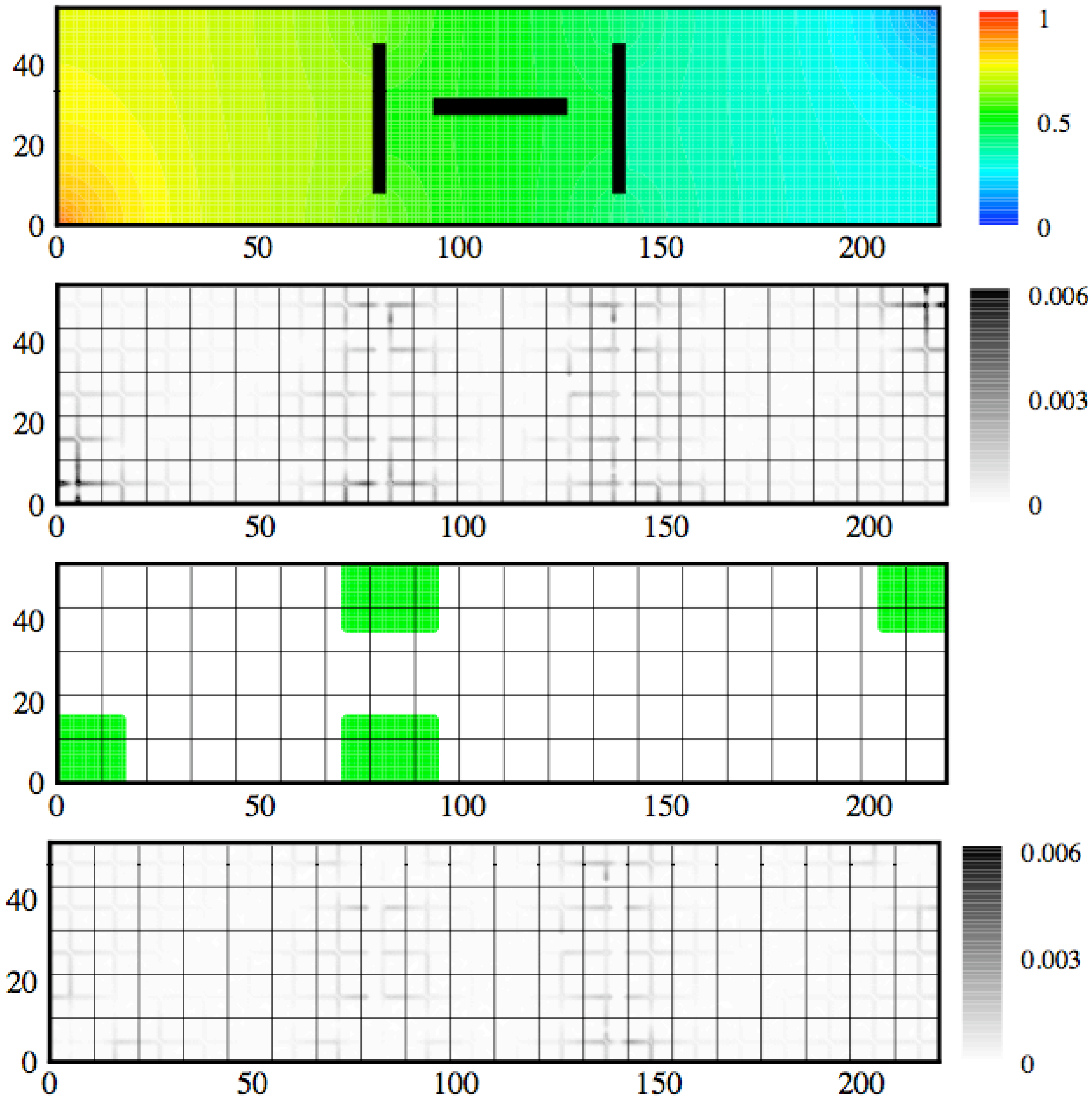




Adaptivity: Residuum



Adaptivity: Residuum



Summary II

- iMSFV method converges even for very difficult cases
- General linear solver/MSFV framework
  - efficient linear solver: cost per iteration is low; #iter indep. of size
  - works for very large aspect ratios (anisotropy)
  - related to DD and AMG, but conservative, even if not converged
  - all advantages of the original MSFV method are preserved
- Solves many problems in an elegant way:
  - anisotropy
  - long structures with sharp contrasts (shale layers)
- Adaptivity:
  - multi-phase solutions can locally be improved
  - even if correction functions are updated only periodically

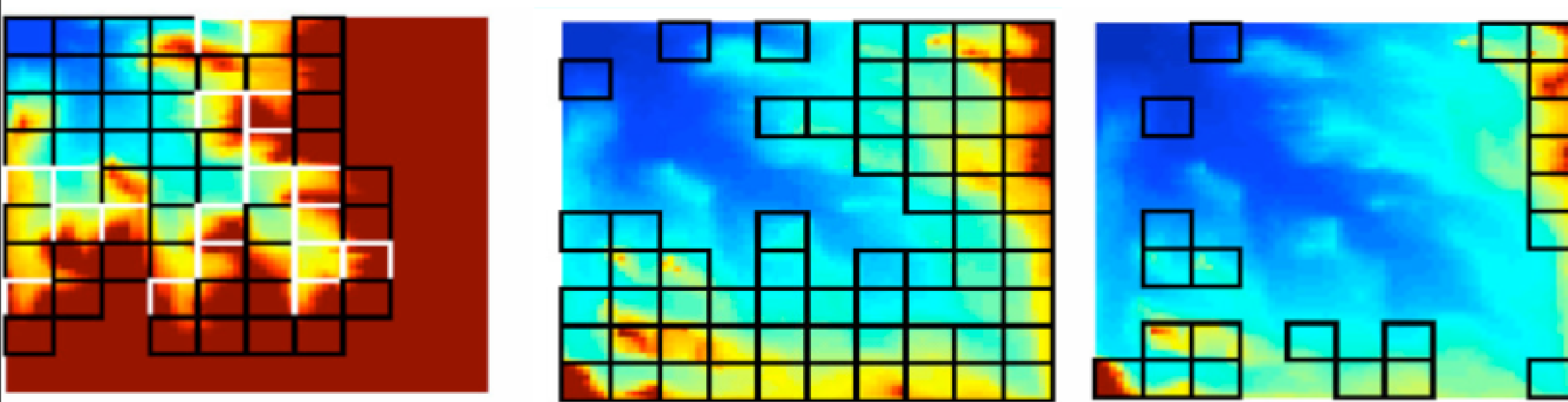
Some Challenges

- General error estimation -> adaptive re-computations
- Multi-level multiscale method
- Automatic optimal choice of coarse variables and decomposition
- General coarsening

- General and efficient implementation for massive parallel computing
- Transport becomes bottel neck

Transport

Adaptive MSFV method (Zhou + Tchelepi)



white cells: fine scale

black cells: coarse scale $\Delta S_i^h = \xi_K^i \Delta S_K^H$

where $\xi_K^i = \frac{\delta S_i^h}{\delta S_K^H}$ for $\Omega_i^h \in \Omega_K^H$