Eidgenössische Technische Hochschule Zürich
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Preconditioning variable viscosity Stokes flow problems associated with a stabilised finite element discretisation

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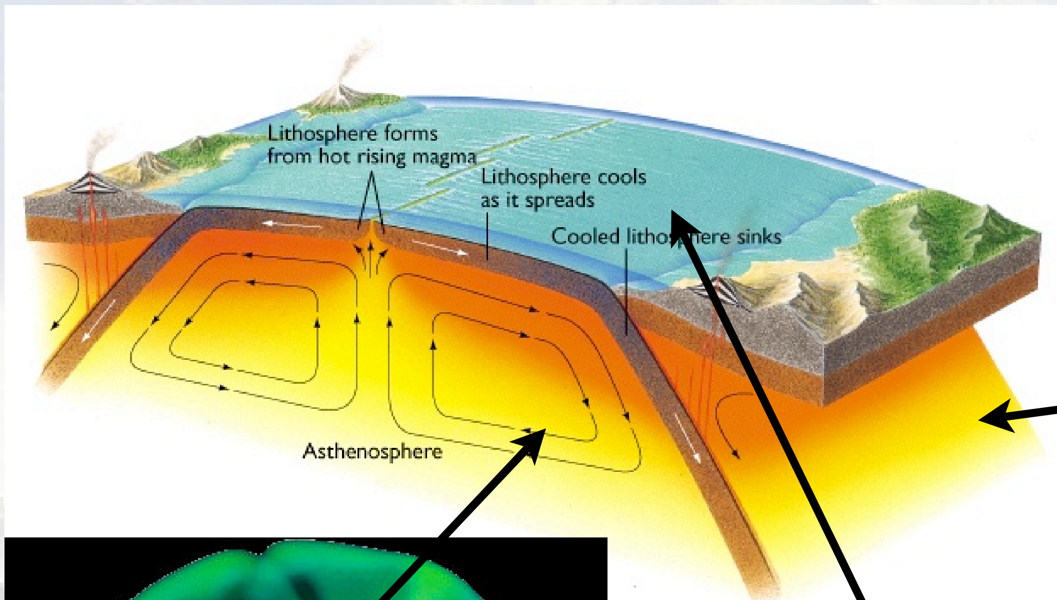
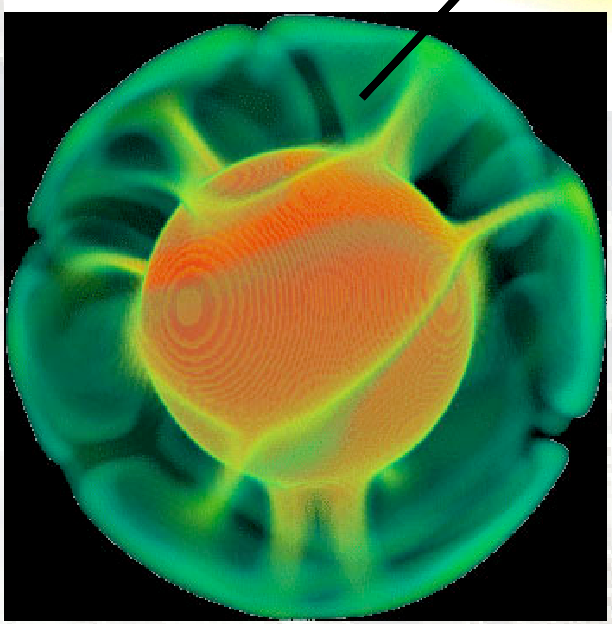
Outline

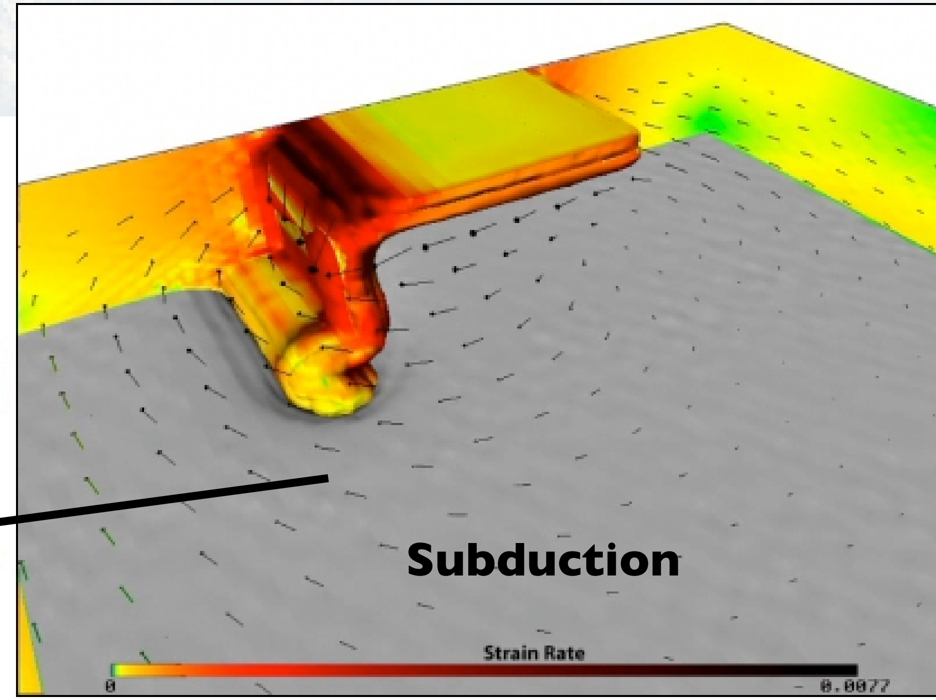
- Motivation
- The stabilised Q1-Q1 element
- Errors
- Preconditioning
- Summary

Motivation

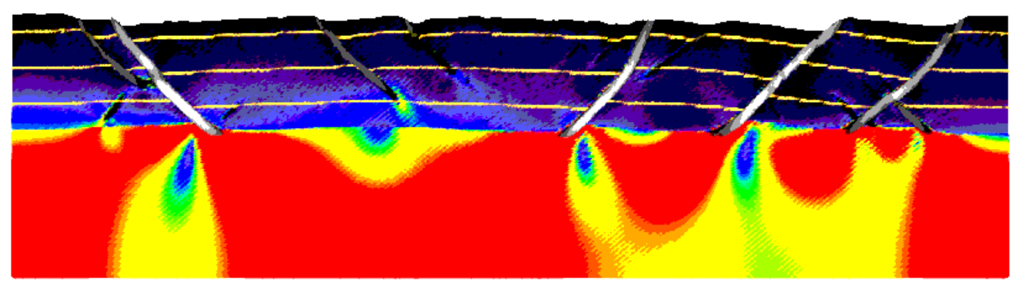




Convection in the mantle



Subduction



Lithospheric deformation

Stokes Flow

- Incompressible Stokes flow with general constitutive tensor.

Momentum $\tau_{ij,j} - p_{,i} + f_i = 0,$

Mass $- u_{i,i} = 0,$

$\Big\}$ in Ω

subject to

i) the boundary conditions

$$u_i = g_i, \qquad \text{on } \Gamma_{g_i}$$
$$\sigma_{ij} n_j = h_i, \qquad \text{on } \Gamma_{h_i},$$

ii) the pressure constraint

$$\int_\Omega p \, dV = p_s,$$ for some constant p_s

Constitutive $\tau_{ij} = \Lambda_{ijkl} \dot{\epsilon}_{kl},$

Strain rate $\dot{\epsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$

We formulate the problem entirely in terms of velocity u, and pressure p.

Stokes Flow + Finite Elements

- Variational problem

$$A(\boldsymbol{v},\boldsymbol{u}) + B(\boldsymbol{v},p) + B(\boldsymbol{u},q) = F(\boldsymbol{v}),$$

$$A(\boldsymbol{v},\boldsymbol{u}) := \int_\Omega 2\eta \sum_{i.j=1}^{d} \dot{\epsilon}_{ij}(\boldsymbol{u})\dot{\epsilon}_{ij}(\boldsymbol{v})\, dV$$

$$B(\boldsymbol{v},p) := -\int_\Omega p\nabla\cdot\boldsymbol{v}\, dV$$

$$F(\boldsymbol{v}) = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{f}\, dV - \int_{\partial\Omega_N} \boldsymbol{v}\cdot\boldsymbol{h}\, dS.$$

$$\begin{pmatrix} K & G \\ G^T & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

Stability issues

Q1-Q1 stabilised

Dohrmann & Bochev, 2004

- Stabilised formulation

$$A(\boldsymbol{v},\boldsymbol{u}) + B(\boldsymbol{v},p) + B(\boldsymbol{u},q) - C(p,q) = F(\boldsymbol{v}),$$

$$C(p,q) := \int_\Omega \frac{1}{\eta}(p - \Pi_0 p)(q - \Pi_0 q)\, dV.$$

Here, PI is L2 projection operator which maps C0 functions onto the space of constant functions.

$$\Pi_0 p^h|_\square = \frac{1}{\|J^e\|}\int_\square p^h\, dV, \qquad \forall K \in \mathcal{T}_h$$

$$\Pi_0 p^h|_\square = \frac{1}{4}\left(p_1^e + p_2^e + p_3^e + p_4^e\right).$$

$$C(p^h,q^h) = \sum_{\square\in\mathcal{T}_h} \frac{1}{\bar{\eta}^e}\int_\square \left(p^h - \Pi_0 p^h\right)|_\square \left(q^h - \Pi_0 q^h\right)|_\square\, dV. \qquad \bar{\eta}^e = \int_\Omega \eta(\boldsymbol{x})\, dV \Big/ \int_\Omega 1\, dV$$

Q1-Q1 stabilised

$$\boldsymbol{C}^e = \frac{1}{\bar{\eta}^e} \int_{\square} (I - \Pi_0)(I - \Pi_0) \|J_e\| dV$$
$$= \frac{1}{\bar{\eta}^e} \left(\boldsymbol{M}^e - \boldsymbol{q}\boldsymbol{q}^T \|J_e\| \right) \qquad \boldsymbol{q} = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})^T$$

$$\begin{pmatrix} K & G \\ G^T & C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

- Benefits
  - Parameter free
  - No macro elements
  - Simple mesh structure can be used
  - Data structure re-use
  - Low order and stable!
- Used by
  - Rhea: *Burstedde et. al., 2008*
  - Underworld: *Moresi et. al., 2008*
- No systematic studies for variable viscosity

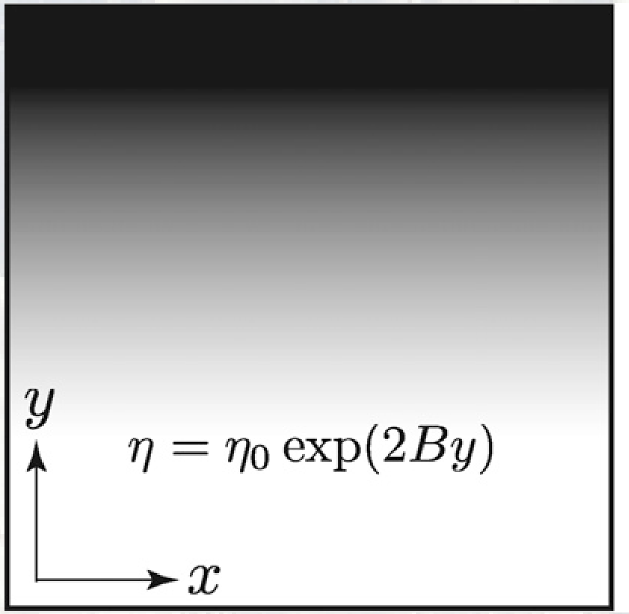
Errors

- Is the discretisation appropriate for variable viscosity flow?
- *Q1-P0* examined by *Moresi et. al., 1996*
- Dohrmann & Bochev error estimates

$$\left(e_u^h\right)_{L_2} = \sqrt{\sum_{i=1}^{d} \int_\Omega \left(u_i^h - u_i\right)^2 \, dV} \qquad \left(e_p^h\right)_{L_2} = \sqrt{\int_\Omega \left(p^h - p\right)^2 \, dV}$$

$$\left(e_u^h\right)_{L_2} = O(h^2) \qquad \left(e_p^h\right)_{L_2} = O(h)$$

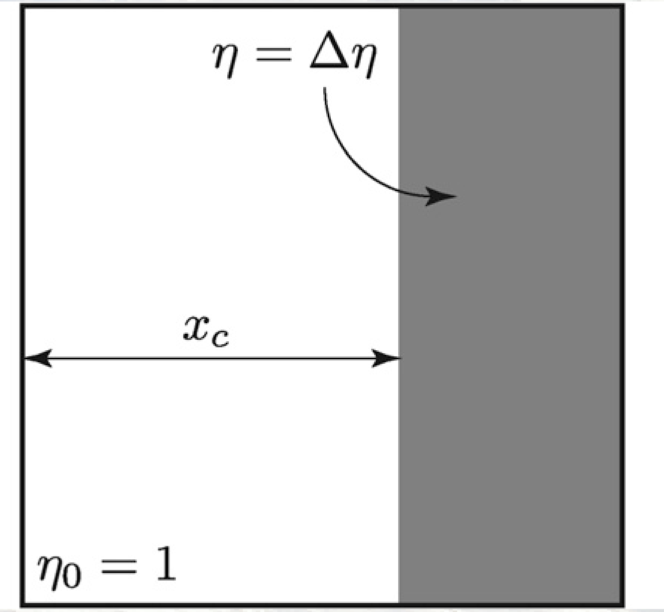
Exponentially varying viscosity: exp(y)



$\Delta\eta$	u	p
10^2	1.99	1.49
10^4	1.97	1.37
10^8	1.91	1.24

(Revenaugh & Parsons, 1987)

Step function viscosity: step(x)



$\Delta\eta$	u	p
10^2	1.91	0.49
10^4	2.00	0.49
10^8	2.00	0.78

(Zhong, 1996)

Finding 1

- Optimal convergence rates for velocity and pressure were obtained for viscosity structures which are continuous.

- For discontinuous viscosity structures, optimal convergence rates were obtained for velocity but sub-optimal convergence rates were obtained for pressure.

Iterative methods for Stokes flow

The ideal approach should be *optimal* in the sense that the convergence rate of method will be bounded independently of

- the discretisation parameters *(Example; grid resolution)*
- the constitutive parameters *(Example; smoothly varying vs. discontinuous viscosity*
- the constitutive behaviour *(Example; isotropic vs. anisotropic)*
- the solution is obtained in *O(n)* time... ie. multigrid

These are a challenging set of requirements

Preconditioners...

- The number of iterations required for convergence is related to the distribution of eigenvalues.
- Preconditioning is the process of "improving" the distribution of eigenvalues, such that the number of iterations is reduced => accelerator

$$Ax = b$$

$$AP^{-1}Px = b \longrightarrow AP^{-1}y = b$$

$$x = P^{-1}y$$

i) $P^{-1} \approx A^{-1}$
ii) should be cheap to construct the preconditioner, P^{-1}
iii) should be cheap to apply the preconditioner, P^{-1}

Schur Complement Reduction

- Decouple u and p
- Solve the Schur complement system

$$\text{solve for } p: \qquad (G^T K^{-1} G - C)p = G^T K^{-1} f - h,$$
$$\text{solve for } u: \qquad Ku = f - Gp.$$

where $S = G^T K^{-1} G - C$ is the Schur complement.

- Represent S as a matrix-free object. To compute $y = Sx$ we

$$\text{compute:} \qquad f^* = Gx,$$
$$\text{solve for } u^*: \qquad Ku^* = f^*,$$
$$\text{compute:} \qquad y = G^T u^* - Cx.$$

- Outer Krylov iterations performed on Sp=h', inner iterations performed on Ky=x.
- Need preconditioners for S and K.

Fully Coupled

- Treat the Stokes problem as single coupled system

$$\begin{pmatrix} K & G \\ G^T & C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \longrightarrow \mathcal{A}x = b, \quad \mathcal{A} \in \mathbb{R}^{(m+n)\times(m+n)}$$

- Apply any suitable Krylov method to $\mathcal{A}x = b$
- We require preconditioner for $\mathcal{A}$.
- Block diagonal or block upper triangular

$$\hat{\mathcal{A}}_d = \begin{pmatrix} \hat{K} & 0 \\ 0 & -\hat{S} \end{pmatrix}, \qquad \hat{\mathcal{A}}_u = \begin{pmatrix} \hat{K} & G \\ 0 & -\hat{S} \end{pmatrix}.$$

Elman, Silvester (1994)
Rusten, Winther (1992)
Silvester, Wathan (1994)

Bramble, Pasciak (1988)
Murphy, Golub (2000)

Both options require preconditioners for K and S

K: Velocity Component Decomposition (VCD)

- Billinear form of the deviatoric stress tensor gradient

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} 2\eta \, \epsilon_{ij}(\boldsymbol{u}) \epsilon_{ij}(\boldsymbol{v}) \, dV$$

- Discrete operator

$$\boldsymbol{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

- A spectrally equivalent billinear form is

[Axelsson, Padiy, "*On a robust and scalable linear elasticity solver based on a saddle point formulation*"]

$$\hat{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \eta \, (\nabla u_k) \cdot (\nabla v_k) \, dV$$

with discrete operator given by

$$\hat{\boldsymbol{K}} = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix}$$

K: Velocity Component Decomposition (VCD)

- Block Gauss-Seidel

$$Ax = b$$

$$x^{k+1} = (D+L)^{-1}\left(b - Ux^k\right)$$

$$A = D + L + U$$

- Discrete counterpart of the stress gradient is given by

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

$$D = \mathrm{diag}\left[K_{11}, K_{22}\right]$$

Mijalkovic & Mihajlovic, 2000

Mihajlovic & Mijalkovic, 2002

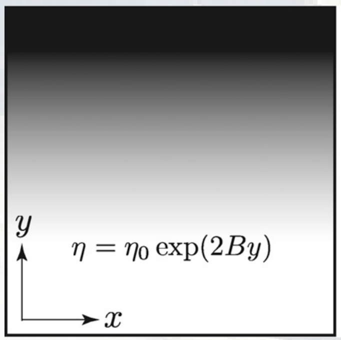
- Treat each velocity component as scalar, variable coefficient diffusion problem.
- Each scalar problem permits effective multigrid preconditioning.

Each K_{ii}^{-1} given by CG, with $\frac{\|r_k\|}{\|r_0\|} < 10^{-2}$, preconditioned via *ML*.

ML: http://software.sandia.gov/Trilinos

Results: exp(y) + step(x)

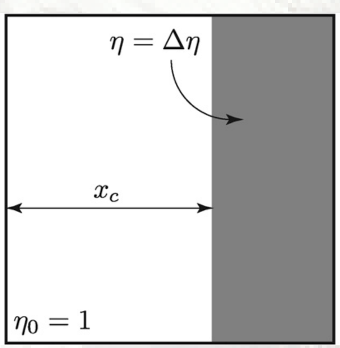
(robustness)



exp(y)

$$\eta = 10^6 \exp(\theta y)$$

elements	θ ($\Delta\eta$)		
$M \times N$	4.6 (10^2)	13.8 (10^6)	18.4 (10^8)
100^2	4	4	4
200^2	4	4	4
300^2	4	4	4

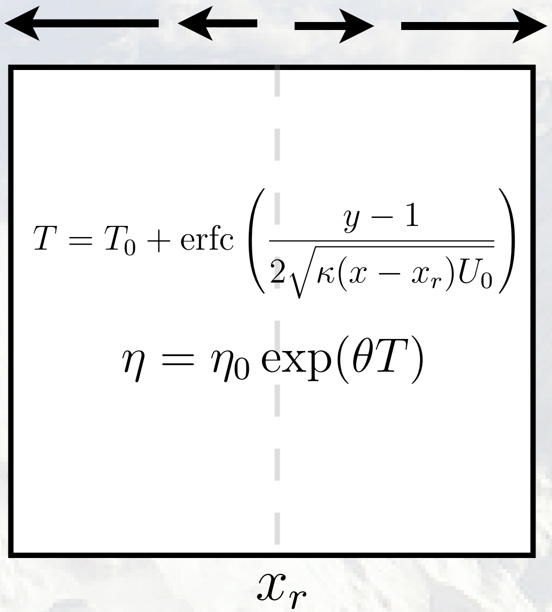


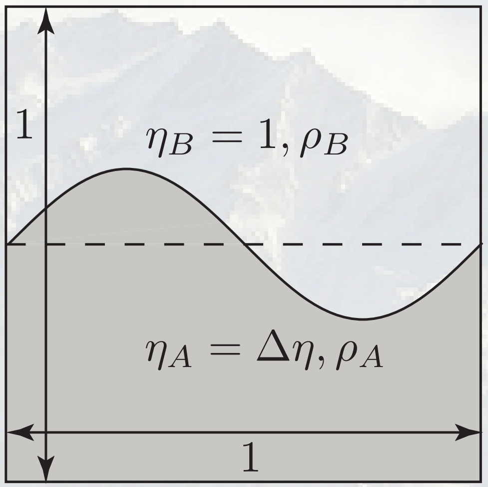
step(x)

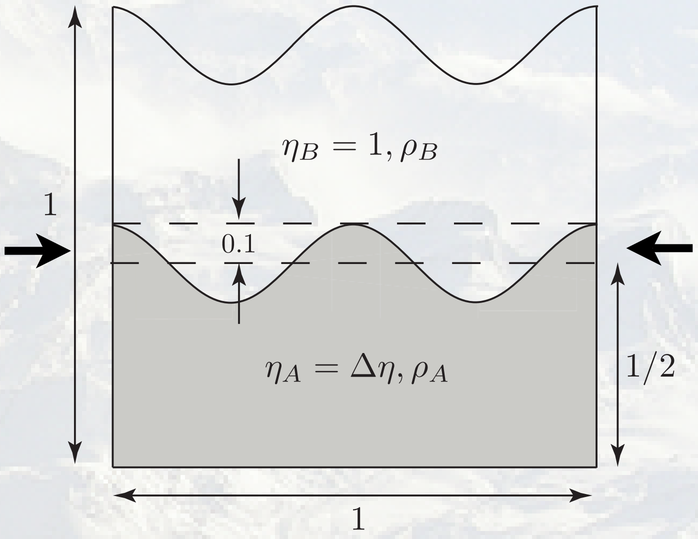
$$\eta = \begin{cases} 1 & x < \frac{1}{2} \\ \Delta\eta & x \geq \frac{1}{2} \end{cases}$$

elements	$\Delta\eta$		
$M \times N$	10^2	10^6	10^{10}
100^2	5	5	5
200^2	5	5	5
300^2	5	5	5

Additional test problems







ridge

- discontinuous / continuous viscosity
- outflow boundaries

diapir

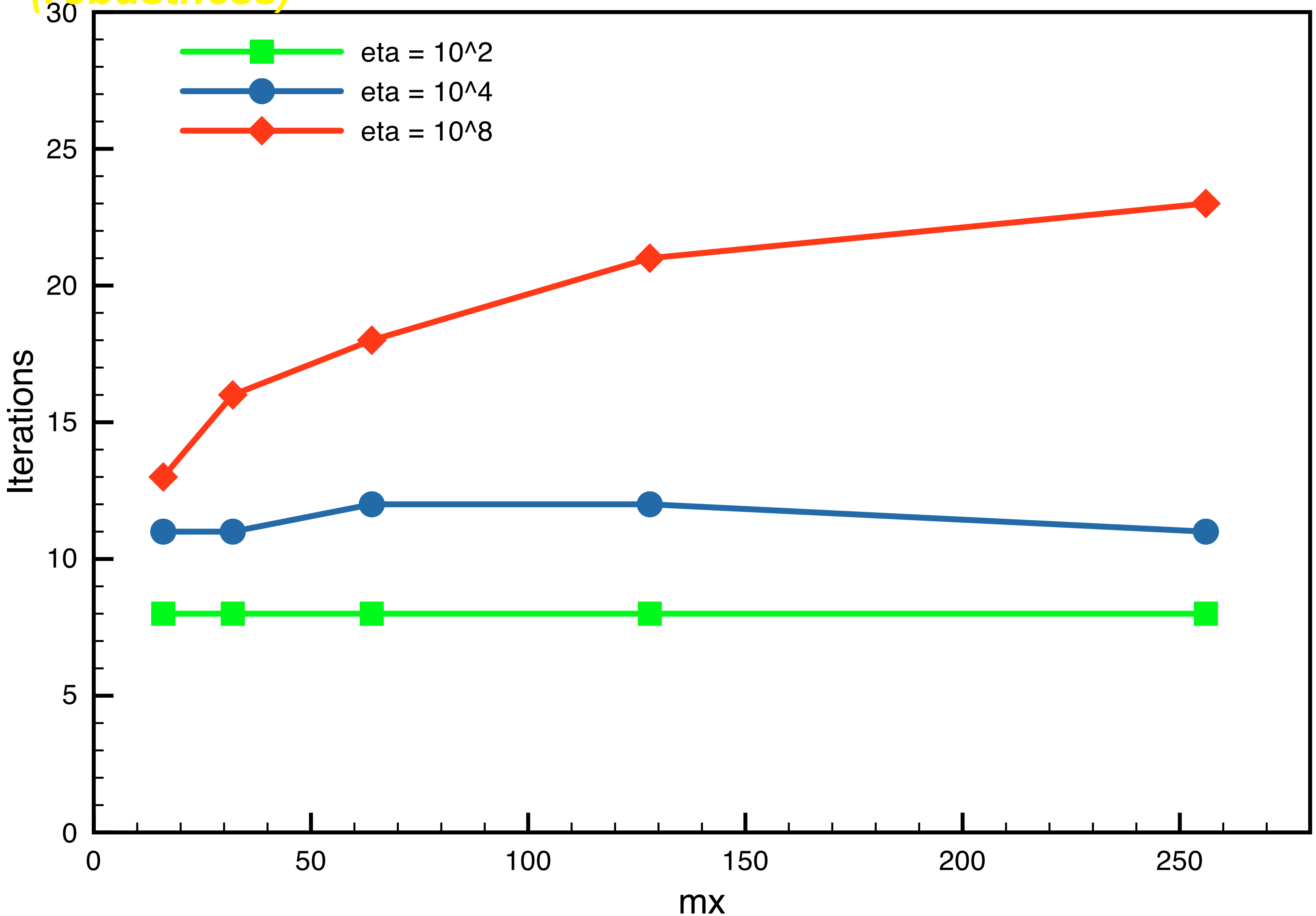
- discontinuous viscosity
- free surface

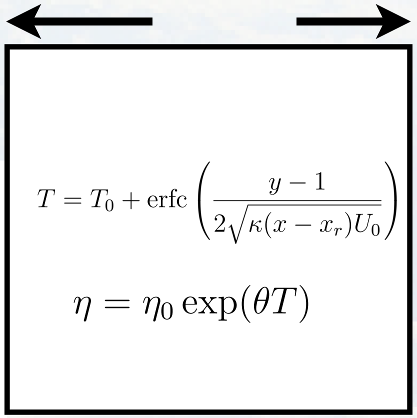
layer

- discontinuous viscosity
- compressional bc's
- free surface
- deformed elements

Results: ridge

(robustness)



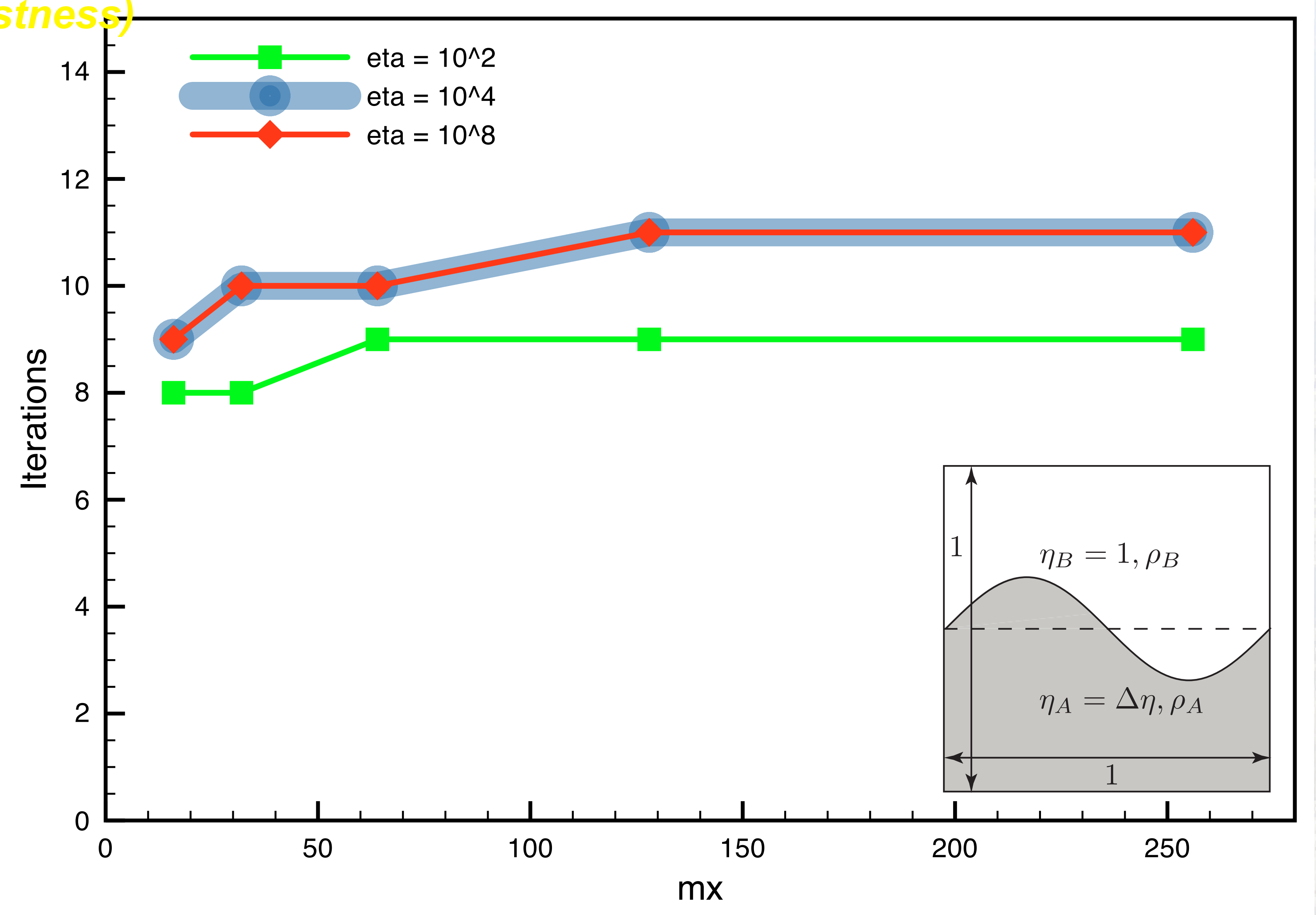


$$T = T_0 + \mathrm{erfc}\left(\frac{y-1}{2\sqrt{\kappa(x-x_r)U_0}}\right)$$

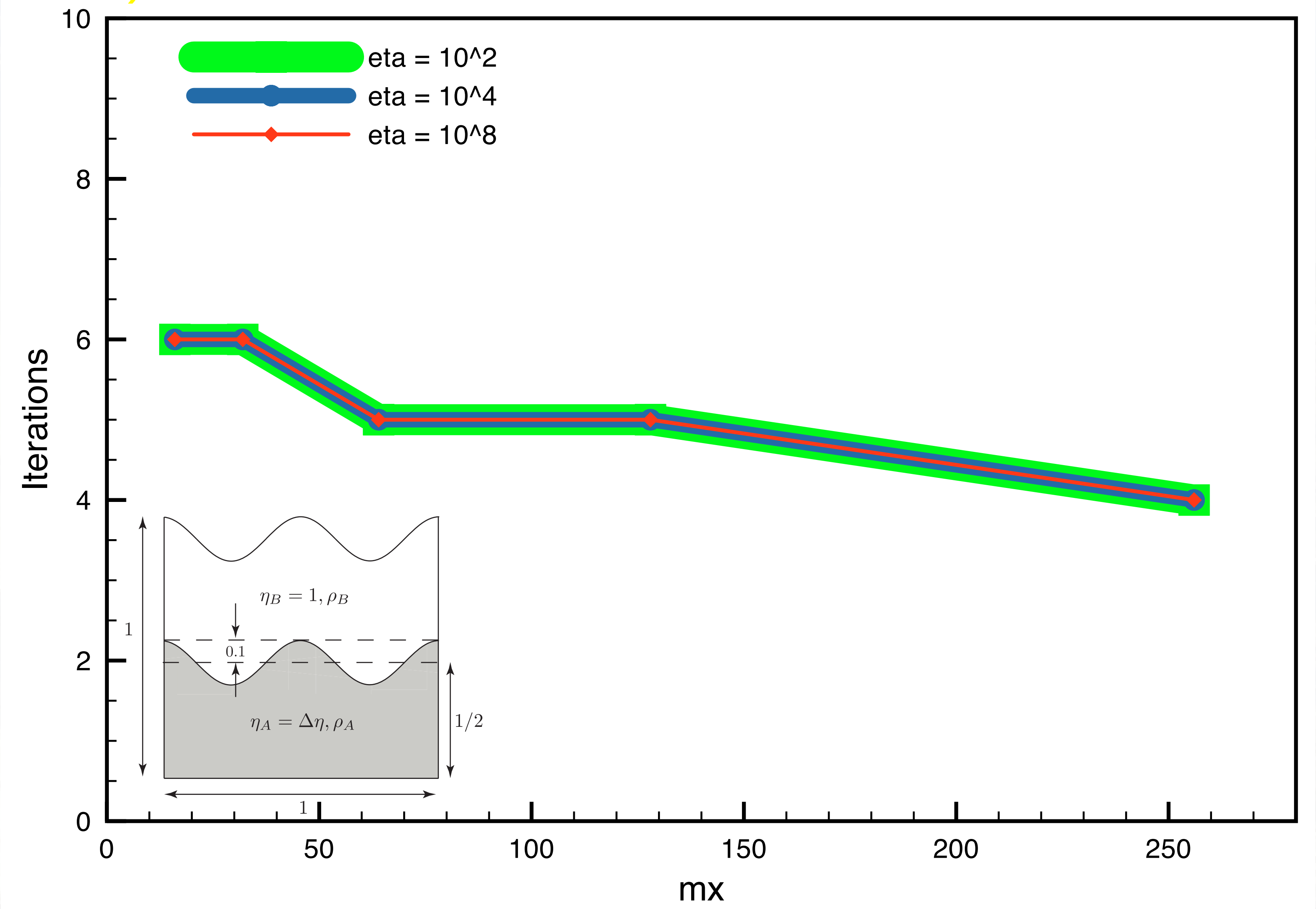
$$\eta = \eta_0 \exp(\theta T)$$

Results: diapir

(robustness)

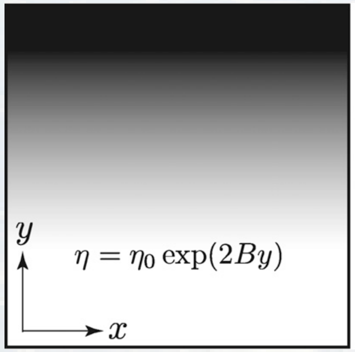


Results: layer
(robustness)



Results: exp(y)

(optimal)



$$\eta = 10^6 \exp(\theta y)$$

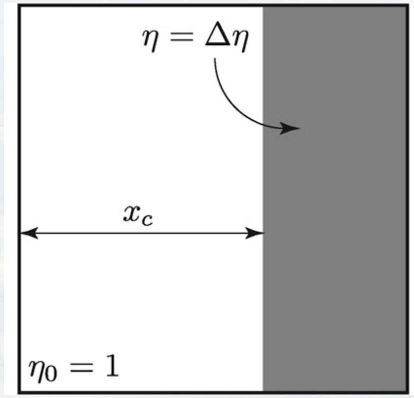
$$\theta = 13.8,\ \Delta\eta = 10^6$$

$M \times N$	unknowns	outer its.	avg. inner its.	CPU time (sec)
200^2	80,802	4	3	1.9
284^2	161,450	4	3	4.3
400^2	321,602	4	3	9.3
566^2	642,978	4	3	19.0
800^2	1,283,202	4	3	37.0
1134^2	2,576,450	4	3	59.6
1602^2	5,139,218	4	3	116.7

Results: step(x)

(optimal)

$$\eta = \begin{cases} 1 & x < \frac{1}{2} \\ 10^6 & x \geq \frac{1}{2} \end{cases}$$



$M \times N$	unknowns	its.	avg. inner its.	CPU time (sec)
200^2	80,802	5	3	1.91
284^2	161,450	5	4	6.00
400^2	321,602	5	4	11.55
566^2	642,978	5	4	27.38
800^2	1,283,202	5	4	50.46

Finding 2

- Block Gauss-Siedel based preconditioner is robust
- Mild dependence on dimension of viscosity structure

- Approach is optimal and maintains robustness, exhibiting *O(n)* solution times when used in conjunction with *ML*
- Parallel efficiency not explored here. Others have demonstrated scalability with *ML.*

Arbenz et. al., 2008
Burstedde et. al., 2008
Thomas Geenan (in yesterdays poster session)

Schur complement (S): ???

- Based on previous ideas...

Scaled BFBt for stabilised systems (compressible)

- Scaled mass matrix

$$\hat{s}_{ij} = -\frac{1}{\bar{\eta}^e} \int_{\Box} M_i M_j \|J_e\| \, dV$$

i) For constant viscosity, the scaled mass matrix is spectrally equivalent to S.

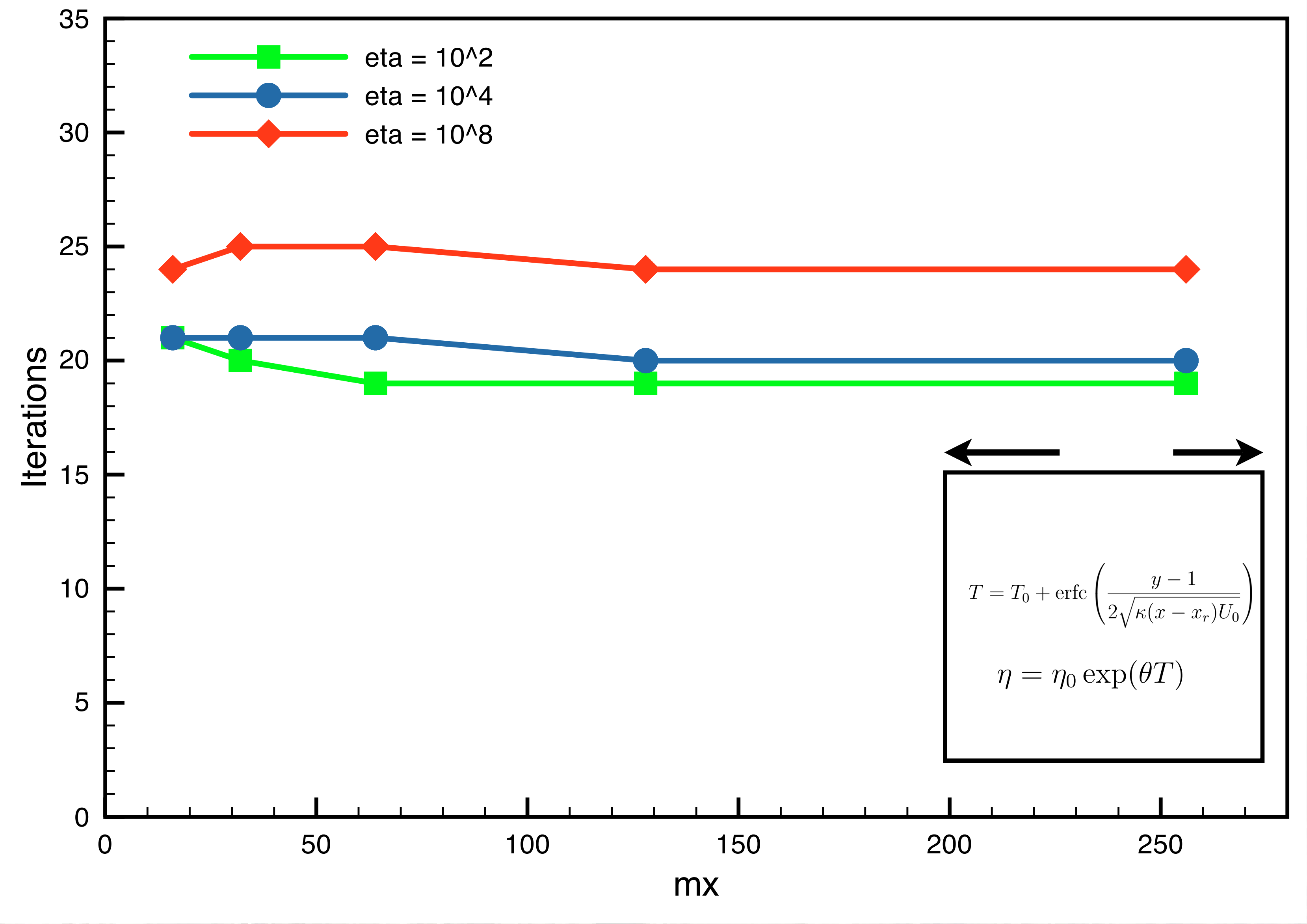
ii) There is a proof that the same result holds for *variable* viscosity.

Thanks Thomas :)

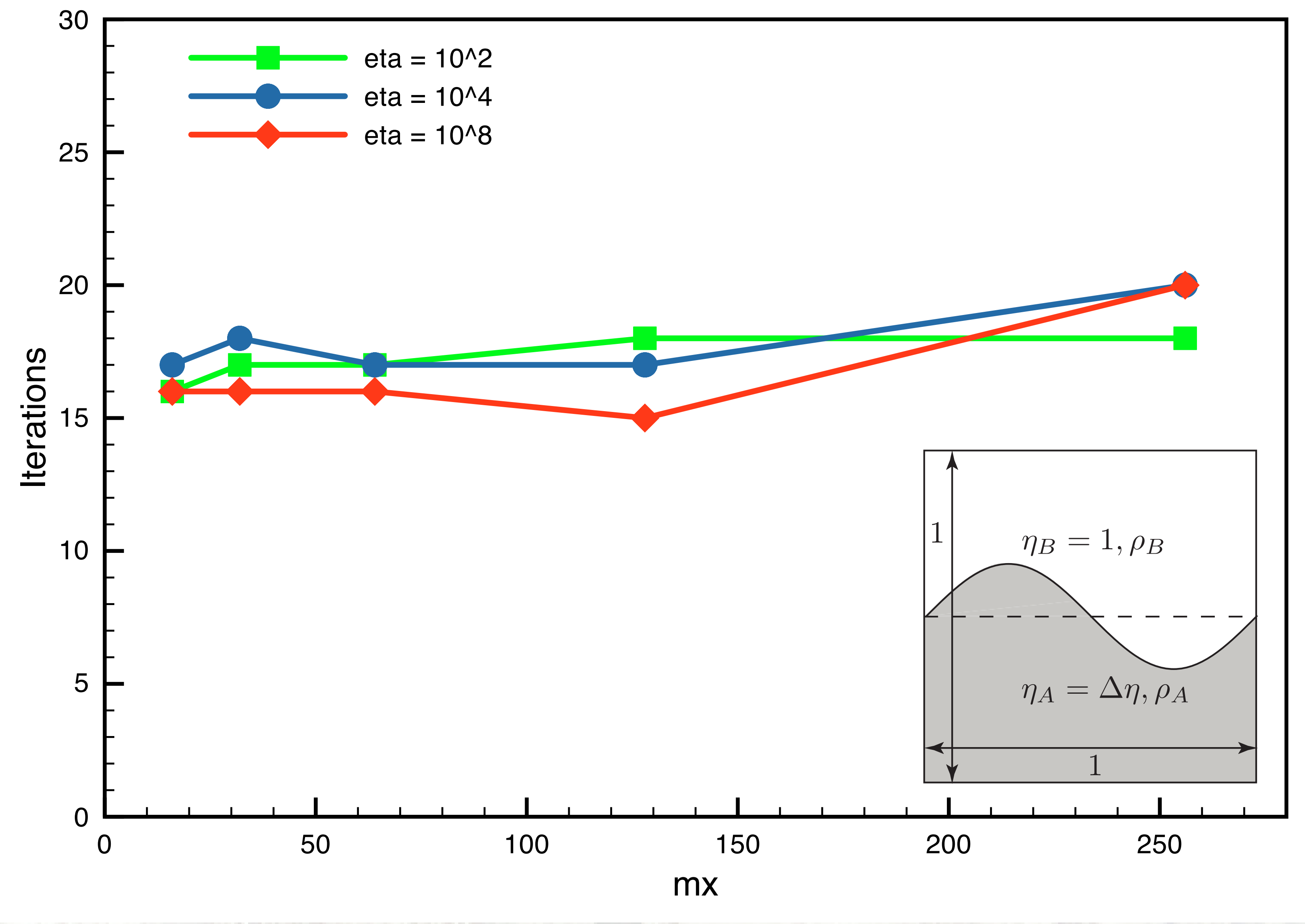
For continuous pressure elements, if the scaled mass matrix is spectrally equivalent to S, then the *diagonal* of the scaled mass matrix is also spectrally equivalent to S *(Elman, 2005).*

We examine these ideas numerically.

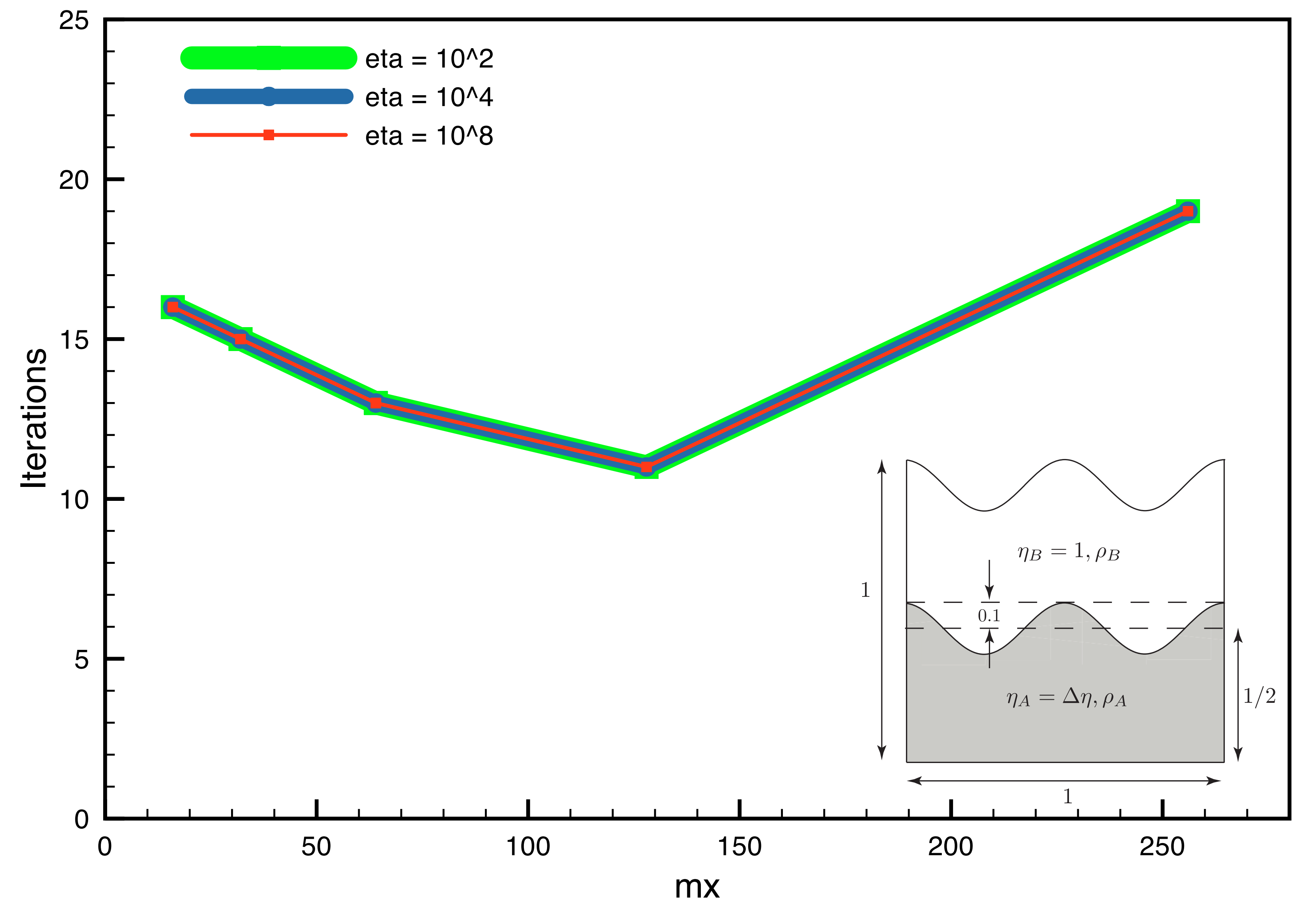
Results: ridge



Results: diapir



Results: layer



Finding 3

- The diagonal of the scaled mass matrix is a surprisingly robust preconditioner, with little dependence on the viscosity contrast.
- The number of iterations required to solve the pressure Schur complement was ≈ 20 for all viscosity structures (ridge, diapir, layer).

Preconditioning strategies

- Putting it all together

$$\begin{pmatrix} K & G \\ G^T & C \end{pmatrix} \begin{pmatrix} X & X \\ X & X \end{pmatrix}^{-1}$$

- Schur complement reduction versus fully coupled approaches?
- Combinations of the two maybe the answer, but the optimal setup seems to be problem dependent.

Summary

- Stabilised Q1-Q1 discretisation is a robust element for studying Stokes flow with large variations in viscosity (smooth or discontinuous).
- Optimal error estimates are preserved for continuous viscosities.
- Velocity component decomposition (VCD) preconditioner is shown to be optimal & robust for variable viscosity Stokes flow.
- The simple diagonal mass matrix preconditioner is a robust and opimal choice for preconditioning the Schur complement when using Q1-Q1 stabilised elements.

Thank-you.... Questions???