# Supplemental Material for: Fluctuation profiles in inhomogeneous fluids

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### Appendix A: Ideal Gas

To illustrate the fluctuation profiles, we consider the ideal gas ($u(\mathbf{r}^N) \equiv 0$) in the presence of an external potential $V_{\rm ext}(\mathbf{r})$. The resulting density profile follows a generalized barometric law [1],

$$\rho_{\rm id}(\mathbf{r}) = \Lambda^{-D} \exp(-\beta(V_{\rm ext}(\mathbf{r}) - \mu)), \tag{A1}$$

where $\Lambda(T) = (2\pi\beta\hbar^2/m)^{1/2}$ indicates the thermal de Broglie wavelength; here $\hbar$ denotes the reduced Planck constant and $m$ is the particle mass. One readily obtains the fluctuation profiles from (1)–(3) [in the main text] as

$$\chi_\mu^{\rm id}(\mathbf{r}) = \beta\rho(\mathbf{r}), \tag{A2}$$

$$\chi_T^{\rm id}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{T}\big(\beta V_{\rm ext}(\mathbf{r}) - \beta\mu + D/2\big), \tag{A3}$$

$$\chi_\star^{\rm id}(\mathbf{r}) = \rho(\mathbf{r}) - \big(\beta V_{\rm ext}(\mathbf{r}) + D/2\big)\rho(\mathbf{r}). \tag{A4}$$

In the context of the OZ relations, (A2) and (A3) are relevant beyond the ideal gas, when $\rho(\mathbf{r})$ is taken to be general. For the ideal case,

$$\rho_{\rm id}(\mathbf{r}) = \Lambda^{-D} \exp\left(\frac{D}{2} - \frac{\chi_T(\mathbf{r})}{k_B\chi_\mu(\mathbf{r})}\right), \tag{A5}$$

$$\chi_\star^{\rm id}(\mathbf{r}) = \frac{1 - D/2}{\Lambda^D} \exp\left(\frac{D}{2} - \frac{\chi_T(\mathbf{r})}{k_B\chi_\mu(\mathbf{r})}\right) - V_{\rm ext}\chi_\mu(\mathbf{r}), \tag{A6}$$

where (A6) follows from (A4) upon using (A2), (A3) and (A5) for $D \neq 2$. The Legendre transform (3) is hence complete, including the replacement of the original variables $T, \mu$ by the new variables $\chi_T, \chi_\mu$. For completeness, the temperature of the inhomogeneous ideal gas can be expressed for $D \neq 2$ via the local susceptibilities as

$$T = (k_B\chi_\mu)^{2/(D-2)} \left(\frac{mk_B}{2\pi\hbar^2}\right)^{D/(2-D)} \exp\left(\frac{2}{2-D}\Big(\frac{D}{2} - \frac{\chi_T}{k_B\chi_\mu}\Big)\right), \tag{A7}$$

where position arguments of $\chi_\mu(\mathbf{r})$ and $\chi_T(\mathbf{r})$ are omitted for clarity. The thermal wavelength satisfies

$$\Lambda^{D-2} = \frac{m}{2\pi\hbar^2\chi_\mu} \exp\left(\frac{D}{2} - \frac{\chi_T}{k_B\chi_\mu}\right). \tag{A8}$$

Throughout, we have retained the temperature dependence of $\Lambda(T)$. Corresponding results for the fluctuation profiles within the frequently used convention of measuring lengths against a molecular size $\sigma$ as the fundamental scale and fixing $\Lambda = \sigma$ from the outset are obtained by formally setting $D = 0$ in (A3)–(A6).